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A Graph Scheme for the Fermion Green's Function Based on the Functional Integral

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Abstract

A recently introduced new diagram technique for the evaluation of Green's functions is extended to the case of Fermi fields.
We use a Hilbert space of anticommuting "classical" variables (so-called Grassmann variables) with involution. The 2-point function is a certain functional integral over this space.
The basic idea is to expand the exponential of the derivative part of the Lagrangean. The resulting sum is treated within a lattice space formalism. The full interaction is contained in some trivial integrals over Grassmann variables. All infinite dimensional integrals are carried out explicitly.
Although the mathematical manipulations are quite different from the usual ones the resulting graphs are very similar to those of the Bose case.
The full graph scheme may be summed explicitly in the case of the free Fermion.
A generalization of the graph scheme to n-point functions is straightforward. At a later stage we hope to treat the Thirring model and the interaction of several fields.
Our formalism is an alternative to the expansion of the exponential of the coupling term which would yield the old-fashioned perturbation theory.

I. Introduction

In this paper we continue our work on the technique of non-perturbative path integral expansions (PIE) for Green's functions. This method was developed in [1] for the 2-point function of a self-interacting Bose field. The generalization to n-point functions is straightforward [2]. The method is based on an approximation of the exponential of the kinetic term of the Lagrangean by a Taylor expansion. This is an extrapolation from the so-called ultralocal static field theories [3] to usual ones. Such an approach is also developed by means of another mathematical method in [4].

After having shown that the method is workable in principle we are interested to develop it further to come near more realistic problems. One step in this direction is the inclusion of spin. Functional integrals over spinor fields were proposed already many years ago [5], [6]. The adequate mathematical tools of handling such functional integrals over anticommuting "classical" fields may be found in [7].

In this paper, we give an introduction to the treatment of self-interacting Fermi fields within our formalism of nonperturbative approximation of Green's functions. We develop a diagram technique very similar to that of the Bose case. All the interaction is contained in so-called f-coefficients. These are combinations of multiple Grassmann integrals over the field functions. The calculations are carried out in a lattice space. After having got all contributions up to a given order the limiting process $\varepsilon \to 0$ is to perform (in general being a difficult problem [1]). As an example we show that the formalism works in the case of the free Fermion.

The paper is organized as follows. Chapter II contains the derivation of the graph scheme within a lattice space. In chapter III the diagram rules are given. The Green's function of the free Fermion is calculated in chapter IV. Some appendixes are given which introduce to lattice space calculations (app. A) and to Grassmann variables (app. B). Appendix C contains something about the M- and f- coefficients, which contain the interaction. In app. D some graphs are evaluated explicitly.

II. Derivation of the graph scheme

We start with the expression for the 2-point function [5], [6]

$$i \tilde{S}_{F\,\alpha\beta}(x-y) = \langle \tilde{0} | T\, \overline{\Psi}_\beta(y)\, \Psi_\alpha(x) | \tilde{0} \rangle = \frac{\frac{1}{\mathcal{N}} \int \mathcal{D}\overline{\Psi}\, \mathcal{D}\Psi\, e^{iS}\, \overline{\Psi}_\beta(y)\, \Psi_\alpha(x)}{\frac{1}{\mathcal{N}} \int \mathcal{D}\overline{\Psi}\, \mathcal{D}\Psi\, e^{iS}} \qquad (2.1)$$

$$S = \int d^dx \left[i \overline{\Psi} \gamma^\mu \partial_\mu \Psi - \mathcal{H}(\overline{\Psi}, \Psi) \right] = \int d^dx \left[\mathcal{L}_0 + \mathcal{L}_i \right] = S_0 + S_i \qquad (2.2)$$

(The conventions are taken from the textbook by Bogoliubov and Shirkov.)

The fields $\Psi_\alpha(x)$ in the functional integral are the so-called classical quantities in the Fermi case, elements of a Grassmann algebra with involution [7]:

$$\{\Psi_\alpha(x), \Psi_\beta(y)\} = \{\Psi_\alpha(x), \Psi_\beta^+(y)\} = \{\Psi_\alpha^+(x), \Psi_\beta^+(y)\} = 0 \qquad (2.3)$$

A useful nomalization is

$$\mathcal{N} = \int \mathcal{D}\overline{\Psi}\, \mathcal{D}\Psi\, e^{iS_i} \qquad (2.4)$$

Our functional integrals are limits of expressions defined in a lattice space with lattice constant ε and volume V:

$$\int \mathcal{D}\overline{\Psi}\, \mathcal{D}\Psi\, e^{iS}\, \overline{\Psi}_\beta(y)\, \Psi_\alpha(x) = \lim_{\varepsilon \to 0} \lim_{V \to \infty} \int \prod_{i,\sigma} d\overline{\Psi}_\sigma(i)\, d\Psi_\sigma(i)\, e^{i\varepsilon^d \sum_j [\mathcal{L}_0(j) + \mathcal{L}_i(j)]}\, \overline{\Psi}_\beta(y)\, \Psi_\alpha(x) \qquad (2.5)$$

with σ : all spinor indexes
  i, j : all cells in the lattice space (with volume ε^d)

Further details on lattice space calculations are given in appendix A, on Grassmann variables in appendix B.
The basic idea is to expand the exponential of the derivative part of the Lagrangean under the functional integral.
For the numerator of (2.1) this reads

$$\sum_{K=0}^{\infty} \frac{1}{K!} \frac{1}{\mathcal{N}} \int \mathcal{D}\overline{\Psi}\, \mathcal{D}\Psi\, e^{iS_i}\, \overline{\Psi}_\beta(y)\, \Psi_\alpha(x)\, [i S_0]^K \qquad (2.6)$$

In the next step we interchange the differentiation with the functional integration. It is convenient for this to rewrite S_o in the following way:

$$i S_o = - \int d^d x \, d^d y \quad G_\mu (x,y) \, \overline{\Psi}(y) \gamma^\mu \psi(x)$$

$$G_\mu (x,y) = \delta(y-x) \frac{\partial}{\partial x^\mu} \tag{2.7}$$

So we get for (2.6)

$$\sum_{K=0}^{\infty} \frac{(-)^K}{K!} \int d^d x_1 d^d y_1 \cdots d^d x_K d^d y_K \quad G_{\mu_1}(x_1,y_1) \cdots G_{\mu_K}(x_K, y_K) \, \varphi_K \tag{2.8}$$

$$\varphi_K = \frac{1}{\mathcal{N}} \int \mathcal{D}\overline{\Psi} \, \mathcal{D}\psi \; e^{i S_i} \; \overline{\Psi}_\beta(y) \, \psi_\alpha(x) \left[\overline{\Psi}(y_1)\gamma^{\mu_1}\psi(x_1)\right] \cdots \left[\overline{\Psi}(y_K)\gamma^{\mu_K}\psi(x_K)\right] \tag{2.9}$$

A corresponding formula results for the denominator.

To be able to evaluate the contributions in all orders we must make some assumptions about $\mathcal{L}_i [\overline{\Psi},\Psi]$:

(i) It must be bilinear in $\overline{\Psi}$ and Ψ .

(ii) It must not be a derivative coupling.

(iii) It must be a local coupling.

Deviations from (ii) or (iii) would require an extension of the formalism.

With (i) to (iii) we are sure that in every space-time cell ψ_σ and $\overline{\Psi}_{\sigma'}$ from the Lagrangean $\mathcal{L}_i$ are located in pairs.

Now we go with φ_K into the lattice space corresponding to (2.5) to perform the functional integration. We make use of the Grassmann integral property (B.10)

$$\int d\overline{\Psi} \, d\psi \quad \overline{\Psi}\psi = -1 \qquad \text{(Other integrals are zero.)}$$

This makes - together with the ψ_σ , $\overline{\Psi}_{\sigma'}$ pairs from S_i - that the fields from $\overline{\Psi}_\beta(y) \, \psi_\alpha(x) \, [\overline{\Psi}\gamma\psi] \cdots [\overline{\Psi}\gamma\psi]$ must be located in the space-time cells in pairs too. This property is independent of the special form of the Lagrangean.

We will give the first φ_K explicitly:

$$\varphi_2 = \frac{\int \mathcal{D}\overline{\Psi}\,\mathcal{D}\Psi\, e^{iS_i}\, \overline{\Psi}_\beta(y)\,\Psi_\alpha(x)}{\int \mathcal{D}\overline{\Psi}\,\mathcal{D}\Psi\, e^{iS_i}}$$

$$= \lim \frac{\int \prod_{i,\sigma} d\overline{\Psi}_\sigma(i)\, d\Psi_\sigma(i)\, e^{i\varepsilon^d \sum_j \mathcal{L}_i(j)}\, \overline{\Psi}_\beta(y)\,\Psi_\alpha(x)}{\int \prod_{i,\sigma} d\overline{\Psi}_\sigma(i)\, d\Psi_\sigma(i)\, e^{i\varepsilon^d \sum_j \mathcal{L}_i(j)}} = \lim_{\substack{\varepsilon\to 0\\ V\to\infty}} \varphi_2^{\varepsilon}$$

$$\varphi_2^{\varepsilon} = \delta_{x,y}\cdot M_{\beta\alpha} \tag{2.10}$$

$$M_{\beta\alpha} = -M_{\alpha\beta} = \frac{\int \prod_\sigma d\overline{\Psi}_\sigma\, d\Psi_\sigma\, e^{i\varepsilon^d \mathcal{L}_i}\, \overline{\Psi}_\beta\,\Psi_\alpha}{\int \prod_\sigma d\overline{\Psi}_\sigma\, d\Psi_\sigma\, e^{i\varepsilon^d \mathcal{L}_i}} \tag{2.11}$$

$M_{\beta\alpha}$ is a trivial multiple integral. The infinity of integrations is absorbed by the adapted normalization of both denominator and numerator.

$$\varphi_4 = \gamma^{M_1}_{\beta_1\alpha_1}\,\frac{1}{\mathcal{N}}\int \mathcal{D}\overline{\Psi}\,\mathcal{D}\Psi\, e^{iS_i}\,\overline{\Psi}_\beta(y)\,\Psi_\alpha(x)\,\overline{\Psi}_{\beta_1}(y_1)\,\Psi_{\alpha_1}(x_1)$$

φ_4 is nonzero only if one of the following relations holds:

$$\begin{array}{ll} x = y = x_1 = y_1: & \delta_{x,y}\,\delta_{x,x_1}\,\delta_{x,y_1} \\ x = y \neq x_1 = y_1: & \delta_{x,y}\,\delta_{x_1,y_1}\,[1-\delta_{x,x_1}] \\ x = y_1 \neq x_1 = y: & \delta_{x,y_1}\,\delta_{x_1,y}\,[1-\delta_{x,x_1}] \end{array}$$

So we get in analogy to φ_2^{ε}:

$$\varphi_4^{\varepsilon} = \gamma^{M_1}_{\beta_1\alpha_1}\Big[\Big(M_{\beta\alpha\beta_1\alpha_1} - (-)^{J(\beta\alpha\beta_1\alpha_1)} M_{\beta\alpha}M_{\beta_1\alpha_1} - (-)^{J(\beta\alpha_1\beta_1\alpha)} M_{\beta\alpha_1}M_{\beta_1\alpha}\Big)$$

$$\delta_{x,y}\,\delta_{x,x_1}\,\delta_{x,y_1}$$

$$+ (-)^{J(\beta\alpha\beta_1\alpha_1)} M_{\beta\alpha}M_{\beta_1\alpha_1}\,\delta_{x,y}\,\delta_{x_1,y_1} + (-)^{J(\beta\alpha_1\beta_1\alpha)} M_{\beta\alpha_1}M_{\beta_1\alpha}\,\delta_{x,y_1}\,\delta_{x_1,y}\Big] \tag{2.12}$$

with

$$M_{\beta\alpha\beta_1\alpha_1} = \frac{\int \prod_\sigma d\overline{\Psi}_\sigma\, d\Psi_\sigma\, e^{i\varepsilon^d \mathcal{L}_i}\,\overline{\Psi}_\beta\,\Psi_\alpha\,\overline{\Psi}_{\beta_1}\,\Psi_{\alpha_1}}{\int \prod_\sigma d\overline{\Psi}_\sigma\, d\Psi_\sigma\, e^{i\varepsilon^d \mathcal{L}_i}} \tag{2.13}$$

Factors like $(-)^{J(\beta\alpha_1\beta_1\alpha)}$ come from the necessary interchanges of $\overline{\Psi}_\beta(y)\ldots\Psi_{\alpha_1}(x_1)$ to separate the two $M_{\beta\alpha_1}$, $M_{\beta_1\alpha}$:

$$\overline{\Psi}_\beta\,\Psi_\alpha\,\overline{\Psi}_{\beta_1}\,\Psi_{\alpha_1} = -\,\overline{\Psi}_\beta\,\Psi_{\alpha_1}\,\overline{\Psi}_{\beta_1}\,\Psi_\alpha$$

In general the φ_K depend on linear combinations of

$$M_{\beta_i\alpha_i\cdots\beta_{i+\varkappa}\alpha_{i+\varkappa}} = \frac{\int\prod_\sigma d\bar\Psi_\sigma\, d\Psi_\sigma\, e^{i\varepsilon^d\mathcal{L}_i}\,\bar\Psi_{\beta_i}\Psi_{\alpha_i}\cdots\Psi_{\alpha_{i+\varkappa}}}{\int\prod_\sigma d\bar\Psi_\sigma\, d\Psi_\sigma\, e^{i\varepsilon^d\mathcal{L}_i}} \tag{2.14}$$

together with factors $\delta_{x_i,y_i}\cdots\delta_{x_i,x_{i+\varkappa}}\,\delta_{x_i,y_{i+\varkappa}}$
and $(-)^{J(\beta_i\cdots\alpha_{i+\varkappa})}$ and $\gamma^{\mu_1}_{\beta_1\alpha_1}\cdots\gamma^{\mu_K}_{\beta_K\alpha_K}$.

The dependence of (2.14) on the spinor indexes must be of the following form:

$$M_{\beta_i\alpha_i\cdots\beta_{i+\varkappa}\alpha_{i+\varkappa}} = M_{2\varkappa}\Big[\sum_{\substack{\text{permutations}\\ \text{of spinor indexes}}}(-)^{j(\beta_j\cdots\alpha_e)}\prod\delta_{\beta_m,\alpha_n}\Big] \tag{2.15}$$

$M_{2\varkappa}$ is some expression depending on $\varkappa$, ε, d and $\mathcal{L}_i$.
One sees immediately from (2.14):

$$M_{\beta_i\alpha_i\cdots\beta_{i+\varkappa}\alpha_{i+\varkappa}} = 0 \quad \text{if} \quad \varkappa > d \tag{2.16}$$

In this case we set $M_{2\varkappa}=0$.
To express the φ_K it is convenient to define new coefficients:

$$f_{\beta\alpha} = \varepsilon^{-d} M_{\beta\alpha} \tag{2.17}$$

$$f_{\beta_1\alpha_1\beta_2\alpha_2} = \varepsilon^{-d}\Big[M_{\beta_1\alpha_1\beta_2\alpha_2} - (-)^{j(\beta_1\alpha_1\beta_2\alpha_2)} M_{\beta_1\alpha_1}M_{\beta_2\alpha_2} - (-)^{j(\beta_1\alpha_2\beta_2\alpha_1)} M_{\beta_1\alpha_2}M_{\beta_2\alpha_1}\Big]$$

$$\vdots$$

The general relation is given in appendix C. There one can also find the coefficients for the free particle case as an example.
The j-coefficients are a short hand notation for

$$j(\beta_j\cdots\alpha_e) = J(\beta_j\cdots\alpha_e) - J(\beta_m\cdots\alpha_n) \tag{2.18}$$

$(\beta_m\cdots\alpha_n)$ is the sequence of the spinor indexes given at the left hand side.
With the new coefficients the expressions φ_K^ε look as follows:

$$\begin{aligned}
\varphi_2^\varepsilon &= \delta_{x,y}\,\varepsilon^d f_{\beta\alpha}\\
\varphi_2^\varepsilon &= \varepsilon^{2d} f_{\beta\alpha}\,\delta_\varepsilon(x-y) \qquad (2.19)\\
\varphi_4^\varepsilon &= \varepsilon^{4d}\gamma^{\mu_1}_{\beta_1\alpha_1}\Big[f_{\beta\alpha\beta_1\alpha_1}\,\delta_\varepsilon(x-y)\,\delta_\varepsilon(x-x_1)\,\delta_\varepsilon(x-y_1)\\
&+(-)^{J(\beta\alpha\beta_1\alpha_1)} f_{\beta\alpha} f_{\beta_1\alpha_1}\,\delta_\varepsilon(x-y)\,\delta_\varepsilon(x_1-y_1) + (-)^{J(\beta\alpha_1\beta_1\alpha)} f_{\beta\alpha_1} f_{\beta_1\alpha}\,\delta_\varepsilon(x-y_1)\,\delta_\varepsilon(x_1-y)\Big]
\end{aligned}$$

$$\vdots$$

Here we have used relation (A.8):

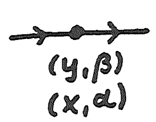
$$\delta_{x,y} = \varepsilon^{d}\,\delta_{\varepsilon}(x-y)$$

Now we have all preliminaries to find the graph scheme which gives us a practical tool to evaluate all the contributions: Every group of equal space-time arguments corresponds to a vertex of a graph. The vertices carry besides the product of $\delta_\varepsilon(x_i-y_j)$ an f-coefficient with corresponding spinor indexes. Every graph has a numerical overall factor, a sign factor from the necessary interchanges of fields and some weight factor (this will be explained later).
Remember the factors $G_{\mu_i}(x_i,y_i)$ in (2.8):

$$G_{\mu_i}(x_i,y_i) = \delta(y_i-x_i)\,\frac{\partial}{\partial x_i^{\mu_i}}$$

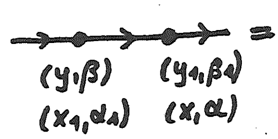
These factors give - unlike the scalar Bose case - a nonsymmetrical connection between the vertex which contains x_i and the vertex which contains y_i. So we define propagators with arrows. Furthermore every propagator includes a γ-matrix and the integration.
Before going into the details we give some examples.

k=0:



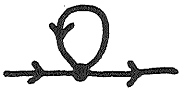
$$= \frac{1}{0!}\,\varepsilon^{(2\cdot 0+2)d}\,f_{\beta\alpha}\,\delta_\varepsilon(x-y) \qquad (2.20)$$

k=1:



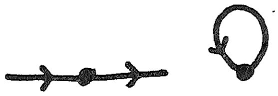
$$= \frac{1}{1!}\,\varepsilon^{(2\cdot 1+2)d}\,(-)^{z(\beta\alpha_1\beta_1\alpha)}\,f_{\beta\alpha_1}\,f_{\beta_1\alpha} \qquad (2.21)$$

$$\gamma^{\mu_1}_{\beta_1\alpha_1}\,(-)^{1}\int d^d x_1\,d^d y_1\,\delta_\varepsilon(y_1-x_1)\,\frac{\partial}{\partial x_1^{\mu_1}}\,\delta_\varepsilon(x-y_1)\,\delta_\varepsilon(x_1-y)$$



$$= \frac{1}{1!}\, \varepsilon^{(2\cdot 1+2)d}\, (-)^{\gamma(\beta\alpha\,\beta_1\alpha_1)}\, f_{\beta\alpha\beta_1\alpha_1} \tag{2.22}$$

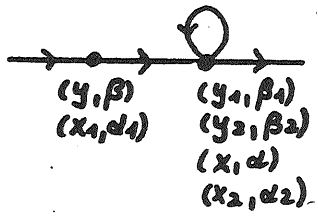
$$\gamma^{\mu_1}_{\beta_1\alpha_1}\, (-)^1 \int d^d x_1\, d^d y_1\, \delta_\varepsilon(y_1-x_1)\, \frac{\partial}{\partial x_1^{\mu_1}}\, \delta_\varepsilon(x-y)\, \delta_\varepsilon(x-y_1)\, \delta_\varepsilon(x-x_1)$$

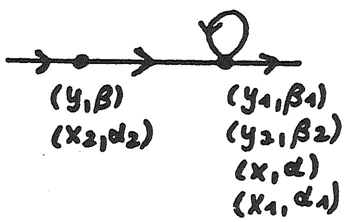


$$= \frac{1}{1!}\, \varepsilon^{(2\cdot 1+2)d}\, (-)^{\gamma(\beta\alpha\,\beta_1\alpha_1)}\, f_{\beta\alpha}\, f_{\beta_1\alpha_1} \tag{2.23}$$

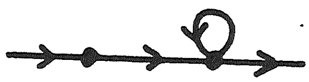
$$\gamma^{\mu_1}_{\beta_1\alpha_1}\, (-)^1 \int d^d x_1\, d^d y_1\, \delta_\varepsilon(y_1-x_1)\, \frac{\partial}{\partial x_1^{\mu_1}}\, \delta_\varepsilon(x-y)\, \delta_\varepsilon(x_1-y_1)$$

Here yet another example for the case k=2; this helps us to understand the origin of weight factors:





These are two contributions which realize our functional integral. But one can prove that they are identical, since they differ only in the arrangement of inner variables. This statement includes all sign conventions. So we calculate only one of these graphs and take account of the other (in general of the others) by a weight factor:



$$= \frac{1}{2!}\, \varepsilon^{(2\cdot 2+2)d}\, \frac{2!}{1!\,1!}\, (-)^{\gamma(\beta\alpha_1\,\beta_1\alpha\,\beta_2\alpha_2)} \tag{2.24}$$

$$f_{\beta\alpha_1}\, f_{\beta_1\alpha\,\beta_2\alpha_2}\, \gamma^{\mu_1}_{\beta_1\alpha_1}\, \gamma^{\mu_2}_{\beta_2\alpha_2}$$

$$(-)^2 \int dx_1 \cdots dy_2\, \delta_\varepsilon(y_1-x_1)\, \delta_\varepsilon(y_2-x_2)\, \frac{\partial}{\partial x_1^{\mu_1}}\, \frac{\partial}{\partial x_2^{\mu_2}}\, \delta_\varepsilon(x_1-y)$$

$$\delta_\varepsilon(x-y_1)\, \delta_\varepsilon(x-x_2)\, \delta_\varepsilon(x-y_2)$$

III. Diagram rules

At first one has to construct all graphs of a given order k, that is all possible nonequivalent graphs consisting of k internal lines with arrows. At each point there must be equal numbers of ingoing and outgoing lines. Every graph for a 2-point function has two free lines.

The rules:

1.
The graph has the overall factor

$$\frac{1}{k!}\,\varepsilon^{(2k+2)d}\,(\text{weight factor}) = \frac{\varepsilon^{(2k+2)d}}{(\text{compensation of multiple counting})} \qquad (3.1)$$

The weight factor depends on the symmetry of the graph. One may interchange all internal lines of a graph - this gives a factor of k! which is compensated by the 1/k! from the Taylor expansion. But the interchange of lines or groups of lines in identical positions (including the direction of arrows) gives no new case. So we have not k! identical graphs but the higher the symmetry the less is the number of equal contributions. This is taken into account by the factor which is called here compensation of multiple counting.

2.
Every vertex carries an f-coefficient with the corresponding number of spinor indexes. All f-coefficients together yield a sign factor $(-)^J$ from the interchange of fields. Formally one can see this by help of the sequence of spinor indexes of the f in comparison to the sequence of the spinor indexes of the γ-matrices.

3.
Every vertex yields the corresponding product of $\delta_\varepsilon(x_i-y_j)$-functions.

4.
Every internal line corresponds to a propagator

$$-\int d^d x_i \; d^d y_i \quad \delta_\varepsilon(x_i - y_i) \, \gamma^{\mu_i}_{\beta_i \alpha_i} \frac{\partial}{\partial x_i{}^{\mu_i}} \tag{3.2}$$

which connects the vertices containing x_i and y_i. The arrow is directed from x_i to y_i.

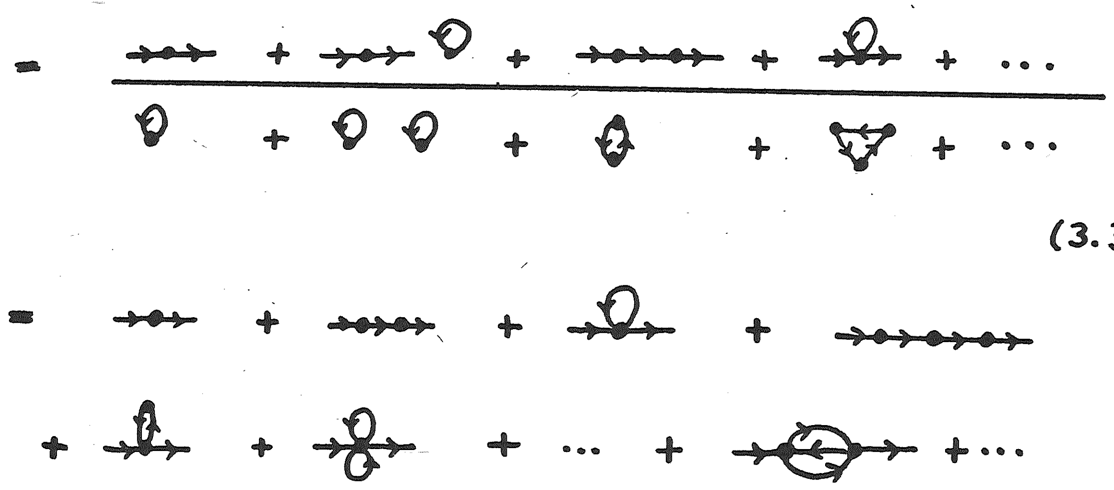
Following these rules one gets an expression which is defined in lattice space (this means the "integrals" are sums, the "derivatives" are differences and so on) and must be reduced, so that the inner variables (x_1, y_1), ... (x_k, y_k) vanish. The resulting expressions look like

$$C_K{}^{\mu_1 \cdots \mu_\varkappa}(\varepsilon) \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_\varkappa}} \delta_\varepsilon(x-y) \; .$$

Some examples are given in appendix D.

Up to now we had under consideration only the numerator of the Green's function (2.1). Here are some remarks about the denominator. While the numerator consists of graphs with two free ends (from $\Psi_\alpha(x)$ and $\overline{\Psi}_\beta(y)$) the denominator consists of graphs without any free end.
The division leeds to the exact cancellation of all disconnected graphs in the numerator with the graphs of the denominator:

$i\tilde{S}_F =$



(3.3)

Since the disconnected graphs are the only ones which depend on the volume V the result is independent of V.
Finally we give the first contributions to the Green's function:

Table 1: The first contributions to $i\tilde{S}_F^{\varepsilon}(x-y)$:

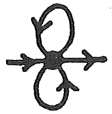
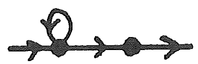
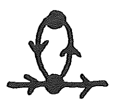
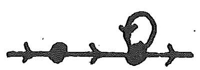
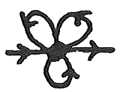
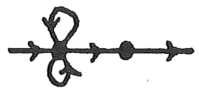
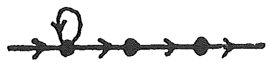
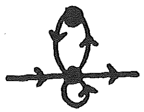
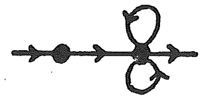
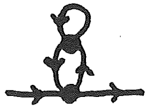
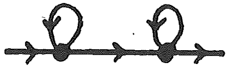
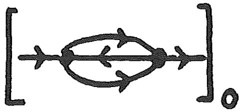
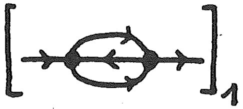
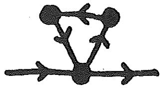
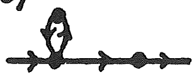
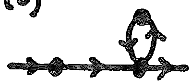
$\delta_\varepsilon(x-y)$	$\frac{\partial}{\partial x^\mu}\,\delta_\varepsilon(x-y)$	$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\,\delta_\varepsilon(x-y)$	$\partial_\mu\partial_\nu\partial_\lambda\,\delta_\varepsilon(x-y)$
F_2 			
$F_4\gamma^\mu\partial_\mu\delta_\varepsilon(0)$ 	$F_2^2\gamma^\mu$ 		
$-\frac{1}{2}(d-2)F_6\,\partial^\mu\delta_\varepsilon(0)\,\partial_\mu\delta_\varepsilon(0)$ 	$F_4F_2\,\gamma^\nu\gamma^\mu\,\partial_\nu\delta_\varepsilon(0)$ 	$F_2^3\gamma^\mu\gamma^\nu$ 	
$F_4F_2[\gamma^\mu\gamma^\nu - d g^{\mu\nu}]\,\partial_{\mu\nu}\delta_\varepsilon(0)$ 	$F_4F_2\,\gamma^\mu\gamma^\nu\,\partial_\nu\delta_\varepsilon(0)$ 		
$-\frac{1}{2}(d-2)F_8\gamma^\mu\partial_\mu\delta_\varepsilon(0)\partial^\nu\delta_\varepsilon(0)\partial_\nu\delta_\varepsilon(0)$ 	$-\frac{1}{2}(d-2)F_6F_2\,\partial^\nu\delta_\varepsilon(0)\,\partial_\nu\delta_\varepsilon(0)\,\gamma^\mu$ 	$F_4F_2^2\gamma^\mu\gamma^\nu\gamma^\lambda\,\partial_\lambda\delta_\varepsilon(0)$ 	$F_2^4\gamma^\mu\gamma^\nu\gamma^\lambda$ 

Table 1 (continued)

$F_6F_2[2\gamma^\mu\gamma^\nu\gamma^\lambda - d\,g^{\lambda\nu}\gamma^\mu]\partial_\mu d_\varepsilon(0)$ $\partial_{\nu\lambda} d_\varepsilon(0)$ 	$-\frac{1}{2}(d-2)F_6F_2\,\partial^\nu d_\varepsilon(0)\,\partial_\nu d_\varepsilon(0)\,\gamma^\mu$ 	$F_4F_2^2\,\gamma^\mu\gamma^\lambda\gamma^\nu\,\partial_\lambda d_\varepsilon(0)$ 	
$F_4^2\gamma^\mu\gamma^\nu\gamma^\lambda\partial_{\mu\lambda} d_\varepsilon(0)\,\partial_\nu d_\varepsilon(0)$ 	$F_4^2\,\gamma^\nu\gamma^\mu\gamma^\lambda\partial_\nu d_\varepsilon(0)\,\partial_\lambda d_\varepsilon(0)$ 	$F_4F_2^2\,\gamma^\lambda\gamma^\mu\gamma^\nu\,\partial_\lambda d_\varepsilon(0)$ 	
$-\frac{1}{3}(d-1)(d-2)F_4^2\,\gamma^\mu 1_\mu\, d_\varepsilon(0) d_\varepsilon^{(3)}(0)$ 	σ 		
$F_4F_2^2\,\gamma^\mu\gamma^\nu\gamma^\lambda\,\partial_{\mu\nu\lambda}\, d_\varepsilon(0)$ 	$-F_4F_2^2[d\,g^{\nu\lambda}\gamma^\mu - \gamma^\nu\gamma^\lambda\gamma^\mu]$ $\partial_{\nu\lambda} d_\varepsilon(0)$ 		
	$-F_4F_2^2[d\,g^{\nu\lambda}\gamma^\mu - \gamma^\mu\gamma^\nu\gamma^\lambda]$ $\partial_{\nu\lambda}\, d_\varepsilon(0)$ 		

Some remarks to table 1:
We have suppressed the indexes α, β for convenience. The F-coefficients are defined in C.8:

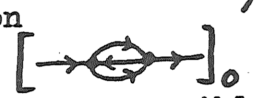
$$F_{2\varkappa} = \varepsilon^{2\varkappa d} f_{2\varkappa}$$

As short hand notations we used

$$\delta_\varepsilon^{(3)}(0) = -3\,\varepsilon^{-d-3}$$

$$\gamma^\mu 1_\mu = \sum_\mu \gamma^\mu g^{\mu\mu}$$

The contribution



is in general not expressible by a term having the original metric. This is a typical deviation from covariance as a consequence of lattice calculations. In the limit $\varepsilon \to 0$ covariance should be reestablished.

IV. Green's function of the free Fermion

In this case the Lagrangean is

$$\mathcal{L}_i = -m\,\overline{\Psi}\Psi \qquad (4.1)$$

The full graph scheme reduces to

$$i\,\tilde{S}_F^{\,\varepsilon}(x,y) =$$  (4.2)

since all $F_\varkappa = 0$ if $\varkappa > 2$.
In appendix D the contribution of order k was determined to be (D.2):

$$g_K = \frac{1}{(-im)^{K+1}}\left[\gamma^{\mu_1}\frac{\partial}{\partial x^{\mu_1}} \cdots \gamma^{\mu_K}\frac{\partial}{\partial x^{\mu_K}}\right]\delta_\varepsilon(x-y) \qquad (4.3)$$

$\lim_{\varepsilon \to 0} g_k$ reduces to $\lim_{\varepsilon \to 0} \delta_\varepsilon(x-y) = \delta(x-y)$, the usual δ-function.
In this trivial case the full series may be summed:

$$\gamma^\mu \partial_\mu \delta(x) = \int \frac{d^dp}{(2\pi)^d}(-i\gamma p)\,e^{-ipx}$$

$$\tilde{S}_{F\,\alpha,\beta}(x,y) = \frac{1}{i}\sum_{K=0}^{\infty} g_K$$

$$= \frac{1}{i}\sum_{K=0}^{\infty}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{-im}\left(\frac{\gamma p}{m}\right)^K e^{-ip(x-y)} = -\int\frac{d^dp}{(2\pi)^d}\,\frac{e^{-ip(x-y)}}{\gamma p - m} \qquad (4.4)$$

so that we have fully reproduced the Green's function for the free Fermion by our method.

Conclusion

We have shown that the PIE method works in the presence of Fermi fields too. There are some technical differences to the Bose case. First of all, we don't perform the functional integration in a complex Hilbert space but take a Grassmann algebra with involution. All integrals become trivial multiple Grassmann integrals which are performed easily. At the other side all manipulations are lenghty because of the variety of spinor indexes. The results are independent of the representation of γ-matrices. The graph scheme is similar to that of the Bose case. A main and not unexpected feature is the presence of propagators with arrows. This gives rise to other weight factors than in the Bose case since propagators are identical (so that their interchange doesn't lead to a new contribution) only if both their end points and their direction of arrow are equal.
A further difference between the graph schemes results from the f-coefficients. Since all $f_{2\varkappa}=0$ if $\varkappa > d$ one must sum in d dimensions only graphs with a maximum of 2d propagators ending in one vertex.
The generalization of the results to n-point functions is straightforward. The next step would be - besides the treatment of the interaction of several fields - the reproduction of the exact solution of the Thirring model. But this demands a generalization of the present formalism which is not yet available.

Appendix A: Calculations in lattice space (d=1)

$$\begin{aligned} f(x) &\rightarrow f(x_l) = f(l\cdot\varepsilon) \\ \partial_x f(x) &\rightarrow \frac{1}{\varepsilon}\left[f(x_l) - f(x_{l-1})\right] \\ \partial_{xx} f(x) &\rightarrow \frac{1}{\varepsilon^2}\left[f(x_{l+1}) - 2f(x_l) + f(x_{l-1})\right] \end{aligned} \tag{A.1}$$

Generally:

$$f^{(2n+1)}(x_l) = \frac{1}{\varepsilon}\left[f^{(2n)}(x_l) - f^{(2n)}(x_{l-1})\right]$$
$$f^{(2n+2)}(x_l) = \frac{1}{\varepsilon}\left[f^{(2n+1)}(x_{l+1}) - f^{(2n+1)}(x_l)\right] \tag{A.2}$$

This gives

$$f^{(K)}(x) = \sum_{l=-\infty}^{+\infty} \gamma_l^{(K)} f(x+l\varepsilon)\,\varepsilon^{-K}$$
$$\gamma_l^{(K)} = \begin{cases} K=2n: & (-)^{n-l}\binom{2n}{n-l} \\ K=2n+1: & (-)^{n-l}\binom{2n+1}{n-l} \end{cases} \tag{A.3}$$

The inverse problem, the lattice analogue to the Taylor expansion, is solved by the following expression:

$$f(x+l\varepsilon) = \sum_{K=0}^{\infty} C_K(l)\, f^{(K)}(x)\,\varepsilon^{K}$$
$$C_K(l) = \begin{cases} K=2n: & \frac{1}{2}\left(1+\frac{n}{l}\right) P_K(|l|) \\ K=2n-1: & \frac{n}{l}\, P_K(|l|) \end{cases} \tag{A.4}$$

The $P_k(|l|)$ are tabulated in [1]:

$$P_K(l) = \begin{cases} \sum_{l'=0}^{|l|-1} \sum_{l''=1}^{l'} P_{K-1}(l''-1) & \text{if } |l| \geq K \\ 0 & \text{else} \end{cases} \tag{A.5}$$

$$P_0(l) = 2$$

The first examples are:

$$\begin{aligned}
f(x) &= f^{(0)}(x) \\
f(x-\varepsilon) &= f^{(0)}(x) - \varepsilon f^{(1)}(x) \\
f(x+\varepsilon) &= f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} \\
f(x-2\varepsilon) &= f^{(0)} - 2\varepsilon f^{(1)} + \varepsilon^2 f^{(2)} - \varepsilon^3 f^{(3)} \\
f(x+2\varepsilon) &= f^{(0)} + 2\varepsilon f^{(1)} + 3\varepsilon^2 f^{(2)} + \varepsilon^3 f^{(3)} + \varepsilon^4 f^{(4)}
\end{aligned} \tag{A.6}$$

Integrations are simulated by sums:

$$\int f(x)\,dx \longrightarrow \varepsilon \sum_l f(x_l) \tag{A.7}$$

$$\delta(x-y) \longrightarrow \delta_\varepsilon(x_l - y) = \frac{\delta_{l,k}}{\varepsilon} \qquad (y = x_k) \tag{A.8}$$

With this scheme we are able to express all derivatives of the δ-function. The usually ill-defined products of derivatives of δ-functions are good expressions within the lattice.
See for example

$$\begin{aligned}
\delta_\varepsilon^{(1)}(x)\ \delta_\varepsilon^{(2)}(x) &= \frac{1}{\varepsilon^2}\left[\delta_{x,0} - \delta_{x-\varepsilon,0}\right] \frac{1}{\varepsilon^3}\left[\delta_{x+\varepsilon,0} - 2\delta_{x,0} + \delta_{x-\varepsilon,0}\right] \\
&= \frac{1}{\varepsilon^5}\left[-2\delta_{x,0} - \delta_{x-\varepsilon,0}\right] \\
&= \frac{1}{\varepsilon^5}\left[-2(\varepsilon\delta_\varepsilon(x)) - (\varepsilon\delta_\varepsilon(x) - \varepsilon^2\delta_\varepsilon^{(1)}(x))\right] \\
&= -3\left[\delta_\varepsilon(0)\right]^4 \delta_\varepsilon(x) + \left[\delta_\varepsilon(0)\right]^3 \delta_\varepsilon^{(1)}(x)
\end{aligned}$$

The generalization of these expressions to d dimensions is straightforward but nontrivial in the details.

Appendix B: Calculations with Grassmann variables

We give only formulae which are used in this paper. For further details see [7].

Let Ψ_n, n= 1, ..., N, be the basic elements of an algebra G_N with the additional property

$$\{\Psi_n, \Psi_m\} = 0 \qquad \text{(B.1)}$$

G_N is called a Grassmann algebra.

On this algebra we define an involution φ :

$$\varphi(\Psi_n) = \Psi_n^+ \qquad (\Psi_n^+ \in G_N^+)$$

with the properties

$$\text{(i)} \qquad \varphi(\Psi_n \Psi_m) = \Psi_m^+ \Psi_n^+ \qquad \text{(B.2)}$$

$$\text{(ii)} \qquad \varphi(\varphi(\Psi_n)) = \varphi(\Psi_n^+) = \Psi_n \qquad \text{(B.3)}$$

$$\text{(iii)} \qquad \varphi(a \Psi_n) = \bar{a} \cdot \Psi_n^+ \qquad (a \ldots \text{complex number}) \qquad \text{(B.4)}$$

The algebra $g_N = G_N \oplus G_N^+$ is the Grassmann algebra with involution used within our paper.

The most general element of g_N is

$$f(\Psi) \equiv f(\Psi_1, \Psi_1^+ \ldots \Psi_N^+) = a_0^0 + a_1^0 \Psi_1 + a_0^1 \Psi_1^+ + \cdots + a_1^1 \Psi_1^+ \Psi_1 + \cdots + a_{12}^0 \Psi_1 \Psi_2 + \cdots + a_{123\cdots N}^{1\cdots N} \Psi_1^+ \Psi_1 \cdots \Psi_N^+ \Psi_N \qquad \text{(B.5)}$$

g_N has the dimension (number of additively independent elements) $2^{N+1}-1$. Even elements of g_N commute with all elements of g_N:

$$[\Psi_n \Psi_m, f(\Psi)] = 0 \qquad \text{(B.6)}$$

All functions of Ψ_n are linear functions:

$$f(\Psi_n) = a_0 + a_n^0 \Psi_n \qquad \text{(B.7)}$$

$$\left.\begin{array}{ll} e^{\Psi_n} = 1 + \Psi_n & e^{\Psi_n + \Psi_m} \neq e^{\Psi_n} e^{\Psi_m} \\ & e^{\Psi_n^+ \Psi_n + \Psi_m^+ \Psi_m} = e^{\Psi_n^+ \Psi_n} e^{\Psi_m^+ \Psi_m} \\ (a + \Psi_n)^2 = a^2 + 2a\Psi_n & \\ \sin \Psi_n = \Psi_n & e^{i\Psi_n} = i \sin \Psi_n + \cos \Psi_n \quad (\cos \Psi_n = 1) \end{array}\right\} \qquad \text{(B.8)}$$

and so on ...

Integrals are defined as linear functionals in the following manner:

$$\{d\Psi_n, d\Psi_m\} = \{d\Psi_n, d\Psi_m^+\} = \{d\Psi_n, \Psi_m\} = \cdots = 0 \tag{B.9}$$

$$\begin{aligned} \int d\Psi_n \Psi_n &= \int d\Psi_n^+ \Psi_n^+ = 1 \\ \int d\Psi_n &= \int d\Psi_n^+ = 0 \end{aligned} \tag{B.10}$$

Multiple integrals must be performed one after the other:

$$\begin{aligned} \int d\Psi_n^+ d\Psi_n e^{\Psi_n^+ \Psi_n} &= \int d\Psi_n^+ d\Psi_n (1 + \Psi_n^+ \Psi_n) = - \int d\Psi_n^+ \Psi_n^+ \int d\Psi_n \Psi_n \\ &= -1 \\ \int d\Psi_n d\Psi_m e^{\Psi_n} e^{\Psi_m} &= \int d\Psi_n d\Psi_m (1+\Psi_n)(1+\Psi_m) = -1 \\ \int d\Psi_n d\Psi_m e^{\Psi_n + \Psi_m} &= \int d\Psi_n d\Psi_m (1 + \Psi_n + \Psi_m) = 0 \end{aligned} \tag{B.11}$$

and so on.

The integral (B.10) is translational and dilatational invariant in analogy to the scalar product in the Bose case:

$$\int d(\Psi + a)\,(\Psi + a) = \int d\Psi\,\Psi \quad \text{since} \quad \int d\Psi\, a = 0 \tag{B.12}$$

$$\int d(a\Psi)\,(a\Psi) = \int (\tfrac{1}{a} d\Psi)(a\Psi) = \int d\Psi\,\Psi \quad \text{since } d(a\Psi) = \tfrac{1}{a} d\Psi \tag{B.13}$$

Appendix C: M- and f-coefficients. Free Fermion coefficients

Derivation of the general relation between M and f:

$$\varphi_K = \gamma^{\mu_1} \cdots \gamma^{\mu_K} \cdot \begin{cases} (-)^\delta M_{\beta\alpha\beta_1\alpha_1\cdots\beta_K\alpha_K} & \text{if } x = y = \cdots = y_K \\ (-)^\delta M_{\beta\alpha} M_{\beta_1\alpha_1\cdots\beta_K\alpha_K} & \text{if } x = y \neq x_1 = \cdots y_K \\ \vdots \ \text{(permutations)} & \\ (-)^\delta M_{\beta\alpha\beta_1\alpha_1} M_{\beta_2\alpha_2\cdots\beta_K\alpha_K} & \text{if } x = y = x_1 = y_1 \\ & \quad \neq x_2 = y_2 = \cdots = y_K \\ \vdots \ \text{(permutations; other cases} & \\ (-)^\delta M_{\beta\alpha} M_{\beta_1\alpha_1} \cdots M_{\beta_K\alpha_K} & \text{if } x = y \neq x_1 = y_1 \\ & \quad \neq \cdots \neq x_K = y_K \\ \vdots \ \text{(permutations)} & \end{cases} \tag{C.1}$$

At the other side we define

$$\varphi_K = \gamma^{\mu_1} \cdots \gamma^{\mu_K} \, \varepsilon^{(2K+2)d} \tag{C.2}$$
$$\cdot \Big[(-)^{j} f_{\beta\alpha\beta_1\alpha_1\cdots\beta_K\alpha_K} \, \delta_\varepsilon(x-y)\,\delta_\varepsilon(x-x_1)\cdots\delta_\varepsilon(x-y_K)$$
$$+ (-)^{j} f_{\beta\alpha} f_{\beta_1\alpha_1\cdots\beta_K\alpha_K} \, \delta_\varepsilon(x-y)\,\delta_\varepsilon(x_1-y_1)\cdots\delta_\varepsilon(x_1-y_K)$$
$$+ \text{ permutations}$$
$$+ \cdots$$
$$+ (-)^{j} f_{\beta\alpha} f_{\beta_1\alpha_1}\cdots f_{\beta_K\alpha_K} \, \delta_\varepsilon(x-y)\,\delta_\varepsilon(x_1-y_1)\cdots\delta_\varepsilon(x_K-y_K)$$
$$+ \text{ permutations} \Big]$$

Setting $x=y=x_1=y_1=\cdots=x_k=y_k$ we get

$$M_{\beta\alpha\beta_1\alpha_1\cdots\beta_K\alpha_K} = \varepsilon^{d} f_{\beta\alpha\beta_1\alpha_1\cdots\beta_K\alpha_K} \tag{C.3}$$
$$+ \varepsilon^{2d} \sum_{\substack{\text{permutations}\\ \text{of } \{\beta\},\{\alpha\}}} (-)^{j(\beta_i\alpha_j\cdots\beta_p\alpha_q)} f_{\beta_i\alpha_j} f_{\beta_m\alpha_n\cdots\beta_p\alpha_q}$$
$$+ \cdots$$
$$+ \varepsilon^{Kd} \sum_{\substack{\text{permut.}\\ \text{of } \{\beta\},\{\alpha\}}} (-)^{j(\beta_i\alpha_j\cdots\beta_p\alpha_q)} f_{\beta_i\alpha_j} f_{\beta_m\alpha_n}\cdots f_{\beta_p\alpha_q}$$

This is the generalization of (2.18).

In analogy to (2.16) is

$$f_{\beta_i\alpha_j\cdots\beta_{i+\varkappa}\alpha_{i+\varkappa}} = f_{2\varkappa} \Big[\sum_{\substack{\text{permutations}\\ \text{of spinor indexes}}} (-)^{j(\beta_m\cdots\alpha_n)} \prod \delta_{\beta_p\alpha_q} \Big] \tag{C.4}$$

Using (2.17) and (C.4) we see:

$$f_{\beta_i\alpha_i\cdots\beta_{i+\varkappa}\alpha_{i+\varkappa}} = 0 \qquad \text{if} \quad \varkappa > d \quad \text{(We set } f_{2\varkappa}=0 \text{ if } \varkappa>d.) \tag{C.5}$$

The only difference between (C.4) and (2.16) is that we have $f_{2\varkappa}$ instead of $M_{2\varkappa}$. Using (C.3) we may find

$$M_{2\varkappa} = \Theta(d-\varkappa) \sum_{\begin{pmatrix}\text{partitions of}\\ \varkappa \text{ into sums of}\\ \text{the form}\\ \sum k_i l_i = \varkappa\end{pmatrix}} \frac{\varkappa!}{\prod (k_i!)^{l_i} \; \prod (l_i!)} \prod \left[\varepsilon^d f_{2k_i}\right]^{l_i} \tag{C.6}$$

$$\Theta(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

This is a formula which was given for M_k in [1] for the Bose case. So we may take the reverse formula from there:

$$f_{2\varkappa} = \Theta(d-\varkappa)\, \varepsilon^{-d} \sum_{\begin{pmatrix}\text{part. of}\\ \varkappa \text{ into sums}\\ \sum k_i l_i = \varkappa\end{pmatrix}} \frac{\varkappa!\, (-)^{l-1} (l-1)!}{\prod (k_i!)^{l_i} \; \prod (l_i!)} \prod \left[M_{2k_i}\right]^{l_i} \tag{C.7}$$

$$l = \sum l_i$$

For later purposes we define modified coefficients

$$F_{2\varkappa} = \varepsilon^{2\varkappa d} f_{2\varkappa} \tag{C.8}$$

The first $f_{2\varkappa}$ are:

$$f_2 = \Theta(d-1)\, \varepsilon^{-d}\, M_2$$

$$f_4 = \Theta(d-2)\, \varepsilon^{-d} \left[M_4 - M_2^2\right]$$

$$f_6 = \Theta(d-3)\, \varepsilon^{-d} \left[M_6 - 3 M_4 M_2 + 2 M_2^3\right]$$

$$f_8 = \Theta(d-4)\, \varepsilon^{-d} \left[M_8 - 4 M_6 M_2 - 3 M_4^2 + 12 M_4 M_2^2 - 6 M_2^4\right]$$

Now we will give the most simple example for the coefficients. Let $\mathcal{L}_i = -m\bar{\Psi}\Psi$ (free particle) and d=1 (only one field component):

$$M_{\beta\alpha} \longrightarrow M_2 = \frac{\int d\bar{\Psi}d\Psi\, e^{-i\varepsilon m\bar{\Psi}\Psi}\,\bar{\Psi}\Psi}{\int d\bar{\Psi}d\Psi\, e^{-i\varepsilon m\bar{\Psi}\Psi}} = \frac{1}{-i\varepsilon m} \qquad (C.9)$$

$$M_{\beta\alpha\beta_1\alpha_1} \longrightarrow M_4 = \frac{\int d\bar{\Psi}d\Psi\, e^{-i\varepsilon m\bar{\Psi}\Psi}\,\bar{\Psi}\Psi\bar{\Psi}\Psi}{\int d\bar{\Psi}d\Psi\, e^{-i\varepsilon m\bar{\Psi}\Psi}} = 0 \qquad (C.10)$$

and so on. If d>1 then

$$M_{\beta\alpha} = \frac{\int d\bar{\Psi}_1 d\Psi_1 \cdots d\bar{\Psi}_d d\Psi_d\, e^{-i\varepsilon^d m(\bar{\Psi}_1\Psi_1+\cdots+\bar{\Psi}_d\Psi_d)}\,\bar{\Psi}_\beta\Psi_\alpha}{\int d\bar{\Psi}_1\cdots d\Psi_d\, e^{-i\varepsilon^d m(\bar{\Psi}_1\Psi_1+\cdots+\bar{\Psi}_d\Psi_d)}}$$

$$= \frac{\int d\bar{\Psi}d\Psi\,\left[1-i\varepsilon^d m\bar{\Psi}\Psi+\frac{(-i\varepsilon^d m)^2}{2!}(\bar{\Psi}\Psi)^2+\cdots\right]\bar{\Psi}_\beta\Psi_\alpha}{\int d\bar{\Psi}d\Psi\,\left[1-i\varepsilon^d m\bar{\Psi}\Psi+\cdots\right]}$$

We must have behind $\int d\bar{\Psi}d\Psi$ (2d different differentials) 2d different fields. In the denominator we get a contribution from

$$\frac{(-i\varepsilon^d m)^d}{d!}\; d!\,\bar{\Psi}_1\Psi_1\cdots\bar{\Psi}_d\Psi_d$$

(d! permutations of the $\bar{\Psi}_\sigma\Psi_\sigma$ give the same contribution.) In the numerator we get a contribution from

$$\frac{(-i\varepsilon^d m)^{d-1}}{(d-1)!}\,(d-1)!\;\bar{\Psi}_1\Psi_1\cdots\bar{\Psi}_d\Psi_d$$

(if $\alpha=\beta$, otherwise zero)

$$M_{\beta\alpha} = M_2\,\delta_{\beta\alpha} \qquad M_2 = \frac{1}{-i\varepsilon^d m}$$

By an analogical calculation

$$M_{\beta\alpha\beta_1\alpha_1} = \theta(d-2)\,M_4\left[\delta_{\beta\alpha}\delta_{\beta_1\alpha_1}-\delta_{\beta\alpha_1}\delta_{\beta_1\alpha}\right] \qquad M_4 = \frac{1}{(-i\varepsilon^d m)^2}$$

and so on:

$$M_{\beta_\nu\alpha_\nu\cdots\beta_{\nu+\varkappa}\alpha_{\nu+\varkappa}} = \frac{\theta(d-\varkappa)}{(-i\varepsilon^d m)^\varkappa}\sum_{\substack{\text{all}\\ \text{permut.}\\ \text{of }\{\beta\}\text{ and }\{\alpha\}}} (-)^{j(\beta_i\alpha_j\cdots\beta_m\alpha_n)}\,\delta_{\beta_i\alpha_j}\delta_{\beta_k\alpha_l}\cdots\delta_{\beta_m\alpha_n} \qquad (C.11)$$

These M-coefficients give rise to the following f-coefficients:

$$f_{\beta\alpha} = \varepsilon^{-d} M_{\beta\alpha} = \frac{1}{-i\varepsilon^{2d} m} \delta_{\beta\alpha} \qquad \text{(C.12)}$$

$$f_{\beta_1\alpha_1\beta_2\alpha_2} = \Theta(d-2)\, \varepsilon^{-d} \Big[M_{\beta_1\alpha_1\beta_2\alpha_2} - (-)^{\gamma} M_{\beta_1\alpha_1} M_{\beta_2\alpha_2} - (-)^{\gamma} M_{\beta_1\alpha_2} M_{\beta_2\alpha_1} \Big]$$

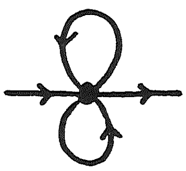
$$= \frac{\Theta(d-2)\,\varepsilon^{-d}}{(-i\varepsilon^{d} m)^2} \Big[(\delta_{\beta_1\alpha_1}\delta_{\beta_2\alpha_2} - \delta_{\beta_1\alpha_2}\delta_{\beta_2\alpha_1}) - (\delta_{\beta_1\alpha_1}\delta_{\beta_2\alpha_2}) + (\delta_{\beta_1\alpha_2}\delta_{\beta_2\alpha_1}) \Big] = 0$$

This result is also true for all the other f-coefficients in this case:

$$f_{\beta_1\alpha_1 \cdots \beta_K\alpha_K} = 0 \quad \text{if} \quad K > 1 \qquad \text{(C.13)}$$

independent of the dimension of the problem.

Appendix D: Some examples of graph evaluation



$$= \varepsilon^{(2\cdot 2+2)d} \frac{1}{2!} f_{\beta\alpha\beta_1\alpha_1\beta_2\alpha_2} (-)^{\gamma(\beta\alpha\beta_1\alpha_1\beta_2\alpha_2)}$$

$$\gamma^{\mu_1}_{\beta_1\alpha_1} \gamma^{\mu_2}_{\beta_2\alpha_2} (-)^2 \int dx_1 \cdots dy_2 \;\; \delta(x_1-y_1)\, \delta(x_2-y_2)$$

$$\frac{\partial}{\partial x_1{}^{\mu_1}} \frac{\partial}{\partial x_2{}^{\mu_2}} \;\; \delta(x-y)\, \delta(x-x_1)\, \delta(x-x_2)\, \delta(x-y_1)\, \delta(x-y_2)$$

Using

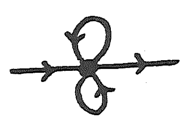
$$(-)^{\gamma(\beta\alpha\beta_1\alpha_1\beta_2\alpha_2)} = +1$$

$$f_{\beta\alpha\beta_1\alpha_1\beta_2\alpha_2} = f_6 \Big[\delta_{\beta\alpha}(\delta_{\beta_1\alpha_1}\delta_{\beta_2\alpha_2} - \delta_{\beta_1\alpha_2}\delta_{\beta_2\alpha_1}) - \delta_{\beta\alpha_1}(\delta_{\beta_1\alpha}\delta_{\beta_2\alpha_2} - \delta_{\beta_1\alpha_2}\delta_{\beta_2\alpha}) - \delta_{\beta\alpha_2}(\delta_{\beta_1\alpha_1}\delta_{\beta_2\alpha} - \delta_{\beta_1\alpha}\delta_{\beta_2\alpha_1}) \Big]$$

one gets with

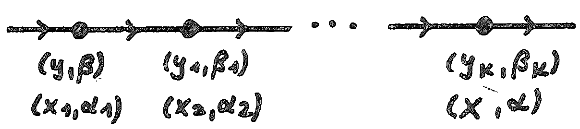
$$\int dx_1 \cdots dy_2 \, [\cdots] \underset{(lattice)}{=} \delta_\varepsilon(x-y) \frac{\partial}{\partial x}{}_{\mu_1} \delta_\varepsilon(0) \frac{\partial}{\partial x}{}_{\mu_2} \delta_\varepsilon(0)$$

and $$\varepsilon^{6d} f_6 = F_6$$



$$= -\frac{1}{2}(d-2)\, F_6 \, \partial^\mu \delta_\varepsilon(0)\, \partial_\mu \delta_\varepsilon(0)\, \delta_{\alpha\beta}\, \delta_\varepsilon(x-y)$$

Another example: free Fermion graph of order k (dimension d)



$$= \varepsilon^{(2k+2)d} (-)^{\mathfrak{J}(\beta\alpha_1\beta_1\alpha_2\beta_2\cdots\beta_k\alpha)} f_{\beta\alpha_1} f_{\beta_1\alpha_2} \cdots f_{\beta_k\alpha}$$

$$\gamma^{\mu_1}_{\beta_1\alpha_1} \cdots \gamma^{\mu_k}_{\beta_k\alpha_k} (-)^k \int d^d x_1 \cdots d^d y_k \; \delta(x_1-y_1) \cdots \delta(x_k-y_k)$$

$$\frac{\partial}{\partial x_1}{}^{\mu_1} \cdots \frac{\partial}{\partial x_k}{}^{\mu_k} \; \delta(y-x_1)\, \delta(y_1-x_2) \cdots \delta(y_k-x)$$

$$= \varepsilon^{(2k+2)d} \frac{1}{(-i\varepsilon^{2d} m)^{k+1}} \delta_{\beta\alpha_1} \delta_{\beta_1\alpha_2} \cdots \delta_{\beta_k\alpha} \gamma^{\mu_1}_{\beta_1\alpha_1} \cdots \gamma^{\mu_k}_{\beta_k\alpha_k}$$

$$\int dx_1 \cdots dx_k \frac{\partial}{\partial x_1}{}^{\mu_1} \cdots \frac{\partial}{\partial x_k}{}^{\mu_k} \; \delta(y-x_1)\, \delta(\dot{x}_1-x_2) \cdots \delta(\dot{x}_k-x)$$

(No derivation with respect to the dotted variables.)

(D.2)

$$= \frac{1}{(-im)^{k+1}} \left[\gamma^{\mu_k} \partial_{\mu_k} \cdots \gamma^{\mu_1} \partial_{\mu_1} \right]_{\alpha\beta} \delta_\varepsilon(x-y)$$

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