Feynman integrals and Mellin-Barnes Representations

– Introduction and Challenges –

Tord Riemann – Silesian University Katowice

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“Case studies of analytical and numerical multiloop methods for future e^+e^- colliders”

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Introduction

Let me cite few words from the talk of Professor van der Bij at the closing meeting of the EU network HEPTOOLS last year, see:

Jochum van der Bij, Acta Phys. Polonica (2017) [4]:

... I mentioned in my congratulory letter to the CERN Director General:

Dear Fabiola,
I want to congratulate CERN for the great running of the machine and the brilliant work of the detectors.
I think the results are great,
I have rarely seen such a convincing null-experiment.
I am sure this will lead to the long overdue paradigm-shift away from the view
“the standard model is wrong and we will have to see what is beyond”
and towards the view
“we know the standard model is true and we have to understand why”.
In the attachment I give you my answer [1-3].
good luck, Jochum
....
6. Conclusion
So now we can conclude. Yes, with the discovery of the Higgs boson scientific history has been made, the results will be lasting. This progress was only made possible through the painstaking hard and precise work by experimental and theoretical physicists, whose work receives little attention in the press or through prizes. We can therefore be grateful to the EU for supporting such research through the “HiggsTools” network and its predecessors “HepTools” and “Physics at Colliders”. Fanciful and science-fiction like scenarios, that have been abundantly with us and in the headlines, ultimately have played little or no role.
With a large circular collider a clear path for the future was sketched, where precision physics will again be crucial.

Precision = Discovery !!

Für Tord Riemann

Du bist zwar ein Preuss

aber hätte ein Friese sein können

Es war mir eine Ehre mit Dir

zu arbeiten

Jochum

Du bist zwar ein Preuss. – aber hätte ein Friese sein können. – Es war mir eine Ehre, mit Dir zu arbeiten. Jochum

You are a Preuss. - But could have been a Frieze. - It was a honor to work with you. Jochum

Contents

MiniWS CERN Jan 2018 [3]

Table 2: Measurement of Electroweak quantities at the FCC-ee, compared with the present precisions.

Observable	Present value	±	error	FCC-ee Stat.	FCC-ee Syst.	Source and dominant exp. error
$\mathrm{m_Z}$ ($\mathrm{keV/c^2}$)	91186700	±	2200	5	100	Z line shape scan Beam energy calibration
Γ_Z (keV)	2495200	±	2300	8	100	Z line shape scan Beam energy calibration
$\mathrm{R}_\ell^\mathrm{Z}$ ($\times 10^3$)	20767	±	25	0.06	1	Ratio of hadrons to leptons Acceptance for leptons
$\alpha_s(\mathrm{m_Z})$ ($\times 10^4$)	1196	±	30	0.1	1.6	$\mathrm{R}_\ell^\mathrm{Z}$ above
$\mathrm{R_b}$ ($\times 10^6$)	216290	±	660	0.3	<60	Ratio of $\mathrm{b\bar{b}}$ to hadrons Stat. extrapol. from SLD [7]
σ^0_had ($\times 10^3$) (nb) $\mathrm{N}_\nu(\times 10^3)$	41541 2991	± ±	37 7	0.1 0.005	4 1	Peak hadronic cross-section Luminosity measurement Z peak cross sections Luminosity measurement
$\sin^2\theta_\mathrm{W}^\mathrm{eff}(\times 10^6)$	231480	±	160	3	2 - 5	$\mathrm{A}_\mathrm{FB}^{\mu\mu}$ at Z peak Beam energy calibration
$1/\alpha_\mathrm{QED}(\mathrm{m_Z})(\times 10^3)$	128952	±	14	4	small	$\mathrm{A}_\mathrm{FB}^{\mu\mu}$ off peak
$\mathrm{A}_\mathrm{FB}^{\mathrm{b},0}$ ($\times 10^4$)	992	±	16	0.02	<1	b-quark asymmetry at Z pole Jet charge
$\mathrm{A}_\mathrm{FB}^{\mathrm{pol},\tau}$ ($\times 10^4$)	1498	±	49	0.15	<2	τ polar. and charge asymm. τ decay physics
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] Beam energy calibration

FCC-ee-Z

$$\frac{\Delta(M_Z)}{M_Z}|_{\mathrm{FCC-ee-Z}} = \frac{\pm 100}{91186700} = 1.09665 * 10^{-6} \tag{1}$$

This is to be compared to the magnitude of electroweak perturbative n-loop orders:

$$\left(\frac{\alpha}{4\pi}\right)^n = 0.000580857^n = \left(6 \times 10^{-4}\right)^n \approx 10^{-3\times n} \tag{2}$$

For the FCC-ee Tera-Z, we need complete electroweak 2-loops and QCD 3-loops, but also enhanced parts of electroweak 3-loops and QCD 4-loops.

The FCC-ee Tera-Z project aims – for many variables of the Z-resonance peak – at a precision

LEP versus FCC-ee-Z

$$\Delta(O_{\mathrm{FCC-ee-Z}}) \approx \frac{1}{20}\ \Delta(O_{\mathrm{LEP}}) \tag{3}$$

Standard references:
A. Blondel, "FCC-ee CDR", in preparation

Blondel,Gluza,Jadach,Janot,TR, hep-ph 2018 [3]
"Standard Model Theory for the FCC-ee: The Tera-Z"
Report on the mini workshop on precision EW and QCD calculations for the FCC studies: methods and techniques, CERN, Geneva, Switzerland, January 12-13, 2018

Feynman integrals at one loop and at higher perturbative orders

Techniques for the calculation of Feynman integrals

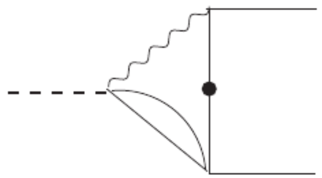
- (Systems of) differential equations
  $\rightarrow$ Kotikov, Laporta, Remiddi, Czakon, Czyz et al.
- (Systems of) difference equations
  $\rightarrow$ Tarasov et al.
- Mellin-Barnes integrals for propagators $1/(p^2 - m^2)$
  $\rightarrow$ Usyukina et al.
- Recurrence relations between Feynman integrals and masters of similar complexity
  $\rightarrow$ Tkachev and Chetyrkin, Laporta, Usovitsch et al. (IBPs), Tarasov et al. (dimensional recurrences)
- Recurrence relations between Feynman integrals and simpler integrals
  $\rightarrow$ Tarasov et al., Dr.Phan, TR, Usovitsch et al.
- Dispersion relations
  $\rightarrow$ Kniehl, Sirlin et al., Cakon, Gluza, TR
- Feynman parameters
  $\rightarrow$ Sector decomposition and Mellin-Barnes representations, and many other techniques
- etc.

Dispersion relations

Bernd Kniehl APP(1996 [5] and refs. therein:
Lectures given at 36th Cracow School of Theoretical Physics, 1996, Zakopane, Poland
“Dispersion relations in loop calculations”

$$F(q^2) = \frac{1}{\pi}\int_{M^2}^{\infty} ds\, \frac{\mathrm{Im}\, F(s)}{s - q^2 - i\epsilon}.$$

A nontrivial application [6]:
Czakon/Gluza/TR, PRD2004
“Master integrals for massive two-loop Bhabha scattering” in QED



V4l3md

$$\texttt{V4l3md} = -\frac{e^{2\epsilon\gamma_E}}{\pi^D}\int \frac{d^Dk_1 d^Dk_2}{[k_2^2 - m^2][\{k_2 - (k_1 + p_1)\}^2 - m^2][k_1^2 - m^2]^2[(k_1 - p_2)^2]}, \tag{4}$$

$$= B_0\left(m^2; \underline{m}^2, 0\right) B_0\left(s; m^2, m^2\right) - \int_{4m^2}^{\infty} \frac{d\sigma}{\pi} \mathrm{Im} B_0(\sigma, m^2, m^2)\left[C_0\left(m^2, s, m^2; \underline{m}^2, 0, \sigma\right) + \frac{B_0\left(m^2; \underline{m}^2, 0\right)}{\sigma - s}\right].$$

The integral is UV and IR divergent.

Feynman integrals and Feynman parameters

Derivations: see e.g. TR, lecture at Computer Algebra School CAPP2017 [7]
Tensor Feynman integrals have the following general form:

$$G(X) = \frac{e^{\epsilon\gamma_E L}}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L \; X(k_{l_1}, \cdots, k_{l_R})}{D_1^{\nu_1} \ldots D_i^{\nu_i} \ldots D_N^{\nu_N}}.$$

The denominator of G:

$$(m^2)^{-(\nu_1+\ldots+\nu_N)} = (x_1 D_1 + \ldots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2Q_j k_j + J)^{-N_\nu} \qquad (5)$$

The scalar Feynman L-loop integral – Symanzik polynomials $U(x), F(x)$

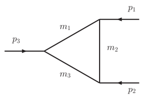
$$G(1) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \, x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

$$\begin{aligned} U(x) &= (\text{det } M) \\ & \quad (\rightarrow 1 \quad \text{for } L = 1) \\ F(x) &= -(\text{det } M)\, J + Q\, \tilde{M}\, Q \\ & \quad \rightarrow -J + Q^2 \quad \text{for } L = 1 \end{aligned}$$

Infrared singularities – The massive 1-loop QED vertex I

Derivations: see e.g. TR, 2017 CAPP Lectures “Feynman diagrams and Mellin-Barnes integrals” [7]

The massive one-loop vertex $C_0(s, m_1, m_2)$



$$C_0 = \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})} \int \frac{d^d k}{[(k+p_1)^2 - m^2][k^2][(k-p_2)^2 - m^2]} \sim |_{k\to\infty} \frac{d^4 k}{k^6} \rightarrow \mathrm{UV-finite}$$

The massive vertex (all $m_1, m_2, m_3 \neq 0$) is a finite quantity.
We assume immediately here: $m_2 = 0, m_1 \neq 0 \neq m_3$.
A problem now is IR-divergence.
Appears when a massive internal line is between two external on-shell lines.

Incoming $p_1^2 = m^2$ and $p_2^2 = m^2$, look at $k \rightarrow 0$:

$$d^4k \frac{1}{(k-p_2)^2 - m^2} \frac{1}{(k)^2} \frac{1}{(k+p_1)^2 - m^2} = d^4k \frac{1}{k^2 - 2kp_2} \frac{1}{(k)^2} \frac{1}{k^2 + 2kp_1} \rightarrow \frac{d^4k}{k^{1+2+1}} \sim \frac{k^3\, dk}{k^4} \sim \frac{dk}{k}|_{k\rightarrow 0} \longrightarrow div$$

An IR-regularization is needed, must take $d > 4$.
Both UV-div (with $d < 4$) and IR-div together:
Must allow for a complex $d = 4 - 2\epsilon$, and take limit at the end.

Have a look at the F-function of the massive QED vertex:

$$\begin{aligned} N &= D_1 x + D_2 y + D_3 z = k^2 x + (k^2 + 2kp_1)y + (k^2 - 2kp_2)z \\ &= k^2 + 2k(p_1 y - p_2 z) = (k+Q)^2 - Q^2 \end{aligned}$$

The F-function is $F = Q^2 - J = Q^2$:

Massive vertex

$$F = [m^2]\,(y+z)^2 + [-s]\,yz \tag{6}$$

This F-function does not factorize in y and z.

Factorize the F-function of the vertex by a variable transformation

Start with change $y \to y' = (1-x)y$, then $y' \to y$:

$$\frac{1}{D_1 D_2 D_3} = \int_0^1 dxdydz \frac{\delta(1-x-y-z)}{(D_2 x + D_1 y + D_3 z)^3} = \int_0^1 dx \int_0^{1-x} \frac{dy}{(D_2 x + D_1 y + D_3 z)^3}$$

$$= \int_0^1 dx \int_0^1 \frac{x dy}{(x^2 \times ([-s]y(1-y) + [m^2]))^3}$$

The integrand factorizes in x and y.

For C_0 we obtain (with $N_\nu = 3$ and $N_\nu - d/2 = 1 + \epsilon$):

QED vertex

$$C_0[s, m, m, 0] = (-1) e^{\epsilon \gamma_E} \Gamma[1+\epsilon] \int_0^1 \frac{dx}{x^{1+2\epsilon}} \int_0^1 \frac{dy}{(p_y^2)^{1-\epsilon}}$$

This is integrable for $\epsilon < 0$, or $d > 4$, or more general: $d \not< 4$.
The x-integral made simple here:

$$\int_0^1 \frac{dx}{x^{1+2\epsilon}} = -\frac{1}{2\epsilon}$$

We see that the IR-singularity is an end-point-singularity in Feynman parameter space.
Now, the usual ϵ-expansion may be performed:

$$-\frac{1}{2\epsilon} \frac{dy}{(p_y^2)^{1-\epsilon}} = -\frac{1}{2\epsilon} \frac{dy}{(p_y^2)} \left[1 + \epsilon \ln(p_y^2) + \epsilon^2 \ln^2(p_y^2) + \cdots \right]$$

Some integrals for 1-loop problems

$$\int dy \ln(y-y_0) = (y-y_0)\ln(y-y_0) - y + C \tag{7}$$

$$\int dy \frac{1}{y-y_0} = \ln(y-y_0) + C \tag{8}$$

$$\int dy \frac{\ln(y-y_0)}{y-y_0} = \frac{1}{2}\ln^2(y-y_0) + C \tag{9}$$

$$\int_0^1 \frac{dx}{x-x_0}\left[\ln(x-x_A) - \ln(x_0-x_A)\right] = Li_2\left(\frac{x_0}{x_0-x_A}\right) - Li_2\left(\frac{x_0-1}{x_0-x_A}\right). \tag{10}$$

There is no such collection beginning at 2-loops.

C_0 with a small photon mass λ

In [8, 9], the C_0-integral is treated with a finite photon mass:

$$\int \frac{d^4k}{(k^2-\lambda^2)(k^2+2kp_1)(k^2-2kp_2)} = = -i\pi^2 \int_0^1 dydx \frac{y}{x^2 p_y^2 + (1-x)\lambda^2}$$
$$= i\pi^2 \int_0^1 dy \left[\frac{1}{2p_y^2} \ln \frac{\lambda^2}{p_y^2} + \mathcal{O}\left(\lambda/\sqrt{p_y^2}\right) \right],$$

It is easy to see from the term $1/(2p_y^2)\ \ln(\lambda^2)$ the correspondence of $(d-4)$ and λ^2, which is a universal relation in all 1-loop cases.

There is no universal relation between $(d-4)$ and λ^2 at higher loop orders.

Sector decomposition

Gudrun Heinrich: Nice review, 2008 [10]

“Sector Decomposition”

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1}$$

Quote:

“The integral contains a singular region where x and y vanish simultaneously, i.e. the singularities in x and y are overlapping.

Our aim is to factorise the singularities for $x \to 0$ and $y \to 0$.”

An answer:

$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \Big(1 + (1-x)\, t\Big)^{-1}$$

$$+ \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \Big(1 + (1-y)\, t\Big)^{-1}$$

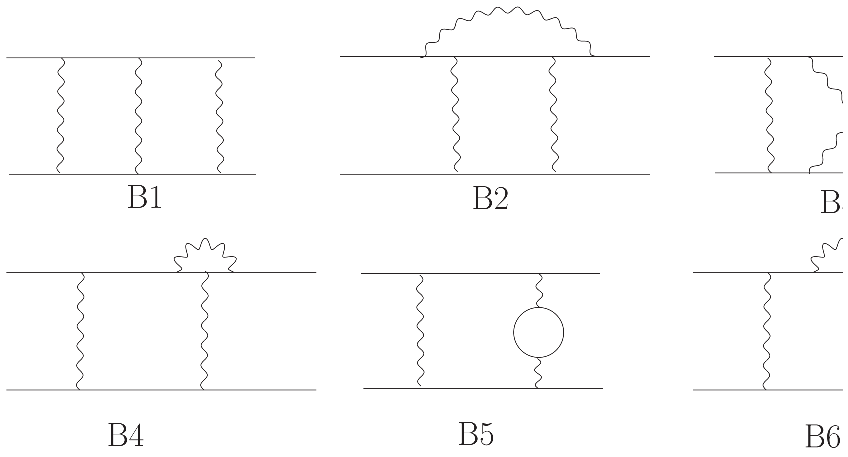


Figure 1: Two-loop box diagrams for massive $2 \to 2$ scattering

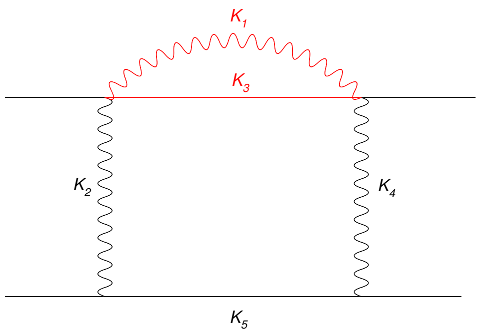


Figure 2: Two-loop box master diagram B5l2m2.
This diagram B5l2m2 is related to B2 = B7l4m2 by shrinking two lines.

Examples for one-loop F-polynomials

One-loop vertex:

$$F(t, m^2) = [m^2](x_1 + x_2)^2 + [-t]x_1x_2$$

one-loop box:

$$F(s, t, m^2) = [m^2](x_1 + x_2)^2 + [-t]x_1x_2 + [-s]x_3x_4$$

one-loop pentagon:

$$F(s, t, t', v_1, v_2, m^2) = [m^2](x_1 + x_3 + x_4)^2 + [-t]x_1x_3 + [-t']x_1x_4 + [-s]x_2x_5 + [-v_1]x_3x_5 + [-v_2]x_2x_4$$

2-loop example:
B7l4m2 = B2 (page 25), has a box-type sub-loop with 2 off-shell legs (see page 25):

$$F^{-(a_{4567}-d/2)} = \left\{[m^2](x_5 + x_6)^2 + [-t]x_4x_7 + [-s]x_5x_6 + [m^2 - Q_1^2]x_7(x_4 + 2x_5 + x_6) + [m^2 - Q_2^2]x_7x_5\right\}^{-(a_{45}}$$

2-loop example:
B5l2m2, has a self-energy sub-loop with off-shell legs (see page 25):

$$F_{2lines}(k_1^2, m^2) = [m^2](x_3)^2 + [-k_1^2 + m^2]x_1x_3$$

Mellin-Barnes integrals

Perform the x-integrations by Mellin-Barnes (MB) integrations

The MBsuite – Computercodes in Mathematica and Fortran/C++

- AMBRE – Derive formal Mellin-Barnes representations for Feynman integrals in $d = 4 - 2\epsilon$: planar and non-planar $L = 2, 3$-loops [11, 12] (2015-2018)
- MB.m – Find (i) a well-defined MB at some ϵ and (ii) a continuation $\epsilon \to 0$ (iii) and then an expansion around $d = 4$; Evaluate numerically in Euclidean regions [13] (2005)
- Further Mathematica codes for simplifying representations and for expansions in a small parameter; Czakon, Kosower, Smirnov [14]
- MBnumerics – Numerical calculation of MB-integrals in Minkowskian space-time [15] (2016-2018)
- CUBA/Cuhre – Numerical integrations using CUBA/Cuhre [16] (2004)

Integrating the Feynman parameters – get MB-Integrals

We derived the examples:

$$SE2l1m = B_0(s,m,0) \quad = \quad e^{\epsilon\gamma_E}\,\Gamma(\epsilon)\int_0^1 dx_1 dx_2 \frac{\delta(1-x_1-x_2)}{F(x)^{\epsilon}} \tag{11}$$

$$V3l2m = C_0(s,m,m,0) \quad = \quad e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{F(x)^{1+\epsilon}} \tag{12}$$

$$F_{SE2l1m} \quad = \quad m^2 x_1^2 + [-s+m^2]x_1 x_2 \tag{13}$$

$$F_{V3l2m} \quad = \quad m^2(x_1+x_2)^2 + [-s]x_1 x_2 \tag{14}$$

We want to do the x-integrations with a general formula, valid for all cases:

$$\int_0^1 \prod_{j=1}^{N} dx_j\, x_j^{\alpha_j-1}\,\delta\left(1-\sum x_i\right) \quad = \quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_N)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_N)} \tag{15}$$

with coefficients α_i dependent on ν_i and on the structure of the F

Eliminate the (+) in (13), (14) $\quad\rightarrow\quad$ For this, apply one or several MB-integrals.

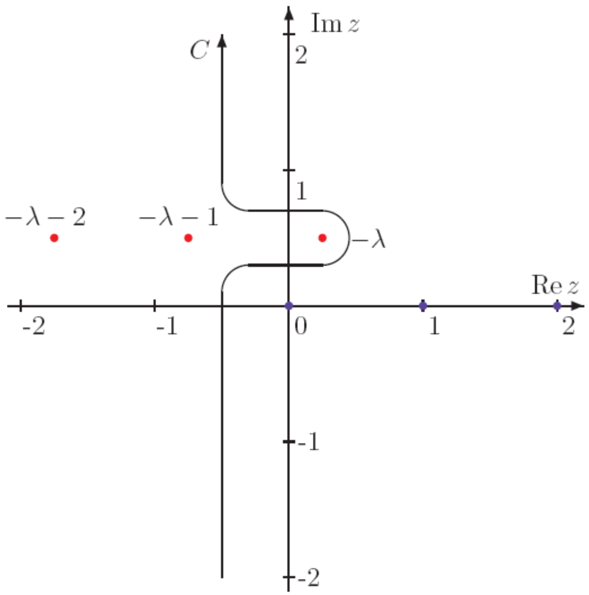
Warning: Will not work out naively for all multi-loop cases: non-planar integrals need special massaging → identify non-planarity (Bielas, Dubovyk et al. 2013) [17]
and use Cheng-Wu-Dubovyk variables, AMBRE v.3 (Dubovyk, Gluza, TR 2015-2018) [2].

The Mellin Barnes representation (Barnes 1908) [18, 19]

Changing the sum $(A+B)$ into a product of A^m and B^n

Derivation: see Whittaker/Watson 1927 [20] and also TR, CAPP lectures

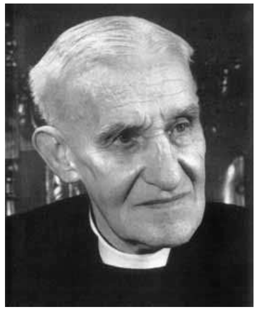
$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}} \tag{16}$$



The integration path separates poles of $\Gamma[\lambda+z]$ and $\Gamma[-z]$.
The formula looks a bit unusual to loop people, but for persons with a mathematical background it is common knowledge.

One might well assume that these two gentlemen did not dream of so heavy use of their results in basic research · · ·

Mellin, Robert, Hjalmar, 1854-1933
Barnes, Ernest, William, 1874-1953



AMBRE and MB and MBsums

How can the Mellin-Barnes formula be made useful in the context of Feynman integrals?

- Apply the MB-relation to propagators and get:

$$\frac{1}{(p^2-m^2)^a} = \frac{1}{2\pi i\,\Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}} \Gamma(a+\sigma)\Gamma(-\sigma) \tag{17}$$

which transforms a massive propagator to an integral over massless ones (with index a of the line changed to $(a+\sigma)$).

- Apply the MB-relation after introduction of Feynman parameters and after the momentum integration to the resulting F- and U-forms, in order to get products of monomials in the x_i, which allows the integration over the x_i:

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i\Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma}\, \Gamma(a+\sigma)\Gamma(-\sigma) \tag{18}$$

Both methods leave Mellin-Barnes integrals to be performed afterwards.

A short remark on history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION" [21]:
A finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals" [22]:
N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box" [23]:
Treat UV and IR divergencies in $d = 4 - 2\epsilon$ by analytical continuations: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", [24]:
Nice algorithmic approach, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined and then going on
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals" [13]:
Tausk's approach realized in an open-source Mathematica program MB.m, numerics good for many Euklidean cases
- Gluza, Dubovyk, Kajda, Riemann, Usovitsch, 2007-2018 $\rightarrow$ AMBRE + MBnumerics [11, 12, 15]:
AMBRE 3 necessary for non-planar diagrams, and MBnumerics necessary in the Minkowskian

The massive QED 1-loop vertex: V3l2m

The Feynman integral V3l2m is the massive QED one-loop vertex function.
It is infrared-divergent.

$$F = [m^2]\ (x_1 + x_2)^2 + [-s]\ x_1 x_2 \tag{19}$$

We will also use $m^2 = 1$ and the the variable

$$y = \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}} \tag{20}$$

Close the integration path to the left, apply the Cauchy theorem – get contributing residua from (and only from) $\Gamma(1+z)$: [25]:

$$\begin{aligned} V(s) &= -\frac{1}{2s\epsilon}\frac{e^{\epsilon\gamma_E}}{2\pi i}\int\limits_{-i\infty-1/2}^{-i\infty-1/2} dz\ (-s)^{-z}\frac{\Gamma^2(-z)\Gamma(-z+\epsilon)\Gamma(1+z)}{\Gamma(-2z)} \\ &= +\frac{e^{\epsilon\gamma_E}}{2\epsilon}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}\frac{\Gamma(n+1+\epsilon)}{\Gamma(n+1)}. \end{aligned} \tag{21}$$

This series may be summed directly with Mathematica![1], and the vertex becomes:

$$\texttt{V3l2m} = V(s) = +\frac{e^{\epsilon\gamma_E}}{2\epsilon}\Gamma(1+\epsilon)\ {}_2F_1\left[1, 1+\epsilon; 3/2; s/4\right]. \tag{22}$$

The vertex is a Gauss hypergeometric function because it is no master – but a tadpole and a self-energy. Vertices are Appell F_1 functions (Tarasov, Dr.Phan, TR et al., 2003-2017) [28, 29, 30]

[1] The expression for $V(s)$ was also derived in [26]; see additionally [27].

Alternatively, one may derive the ϵ-expansion by exploiting the well-known relation with harmonic numbers $S_k(n) = \sum_{i=1}^{n} 1/i^k$ in (21):

$$\frac{\Gamma(n+a\epsilon)}{\Gamma(n)} = \Gamma(1+a\epsilon)\exp\left[-\sum_{k=1}^{\infty}\frac{(-a\epsilon)^k}{k}S_k(n-1)\right]. \tag{23}$$

The product $\exp(\epsilon\gamma_E)\Gamma(1+\epsilon) = 1+\frac{1}{2}\zeta[2]\epsilon^2+O(\epsilon^3)$ yields expressions with zeta numbers $\zeta[n]$, and, taking all terms together, one gets a collection of inverse binomial sums[2]; the first of them is the IR divergent part:

$$V(s) = \frac{V_{-1}(s)}{\epsilon} + V_0(s) + \cdots \tag{24}$$

$$V_{-1}(s) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{1}{2}\frac{4\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}} = \frac{y}{y^2-1}\ln(y). \tag{25}$$

[2] For the first four terms of the ϵ-expansion in terms of inverse binomial sums or of polylogarithmic functions, see [25].

There is also the opportunity to evaluate the MB-integrals numerically by following with e.g. a Fortran routine the straight contour.
This applies after the ϵ-expansion.

$$\int_{-5i+\Re e(z)}^{+5i+\Re e(z)}$$

is usually sufficient.
This numerical approach works fast and stable for Euclidean kinematics where $-s > 0$.

MBsums

Mathematica package for deriving multi-sums from MB-representations

Michal Ochman made a Mathematica package for the automated derivation of multiple sums from Mellin-Barnes representations.

Seems to work fine, is in use sometimes, but can be made much more efficient.

The implementation in MBsums looks not practicable already for 4-fold integrals.
Ochman, TR 2015 [31, 32, 33]

Residing problem: Sum up the sums

Expansion in a small parameter: vertex V3l2m for m^2/s

Use as an example for determining the small mass expansion:

$$\text{V3coefm1} = \text{Coefficient}[\text{V3l2m}[[1,1]],\epsilon,-1] \tag{26}$$

$$= -\frac{1}{2s}\frac{1}{2\pi i}\int_{-i\infty-1/2}^{+i\infty-1/2} dz \left(-\frac{m^2}{s}\right)^z \frac{\Gamma_1[-z]^3\Gamma_2[1+z]}{\Gamma_3[-2z]} \tag{27}$$

If $|m^2/s| << 1$, then the smallest [positive] power of it gives the biggest contribution: its exponent has to be positive and small. So, close the contour to the right (positive z), and leading terms come from the residua expansion due to poles of $\Gamma_1[-z]^3$ at $z = -1, -2, \cdots$. The residues are terms of a binomial sum:

$$Residue[n] = +\frac{1}{s}\left(\frac{m^2}{s}\right)^n \frac{(2n)!}{(n!)^2}\left[2HarmonicNumber[n] - 2HarmonicNumber[2n] - \ln\left(-\frac{m^2}{s}\right)\right]$$

with first terms equal to (-1)*Residua:

$$V3l2m = \frac{1}{s}\left[\ln\left(-\frac{m^2}{s}\right) + \frac{m^2}{s}\left(2 + 2\ln\left(-\frac{m^2}{s}\right)\right) + \frac{m^4}{s^2}\left(7 + 6\ln\left(-\frac{m^2}{s}\right)\right)\right] + O(m^6/s^4) \tag{28}$$

MBnumerics, Johann Usovitsch (plus a bit by I.D. + TR)

Mathematica package for numerical calculation of MB-integrals in Minkowskian kinematics

Example massive QED 1-loop vertex

Usovitsch et al. PoS(LL2016) [35]

Contour deformations

$$V_{-1}(s) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{2\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}}.$$

$$z = \Re[z] + i\,t, \quad t \in (-\infty, +\infty) \tag{3.5}$$

where $\Re[z]$ is chosen in three different ways (see Fig. 1):

$$z(t) = x_0 + it \;:\quad V_{-1}^{C_1}(s) = \int_{-\infty}^{+\infty} (i)\, dt\, J[z(t)], \tag{3.6}$$

$$z(t) = x_0 + \theta t + it \;:\quad V_{-1}^{C_2}(s) = \int_{-\infty}^{+\infty} (\theta + i)\, dt\, J[z(t)], \tag{3.7}$$

$$z(t) = x_0 + at^2 + it \;:\quad V_{-1}^{C_3}(s) = \int_{-\infty}^{+\infty} (2at + i)\, dt\, J[z(t)]. \tag{3.8}$$

Numerical integration of massive two-loop Mellin-Barnes integrals in Minkowskian regions Janusz Gluza

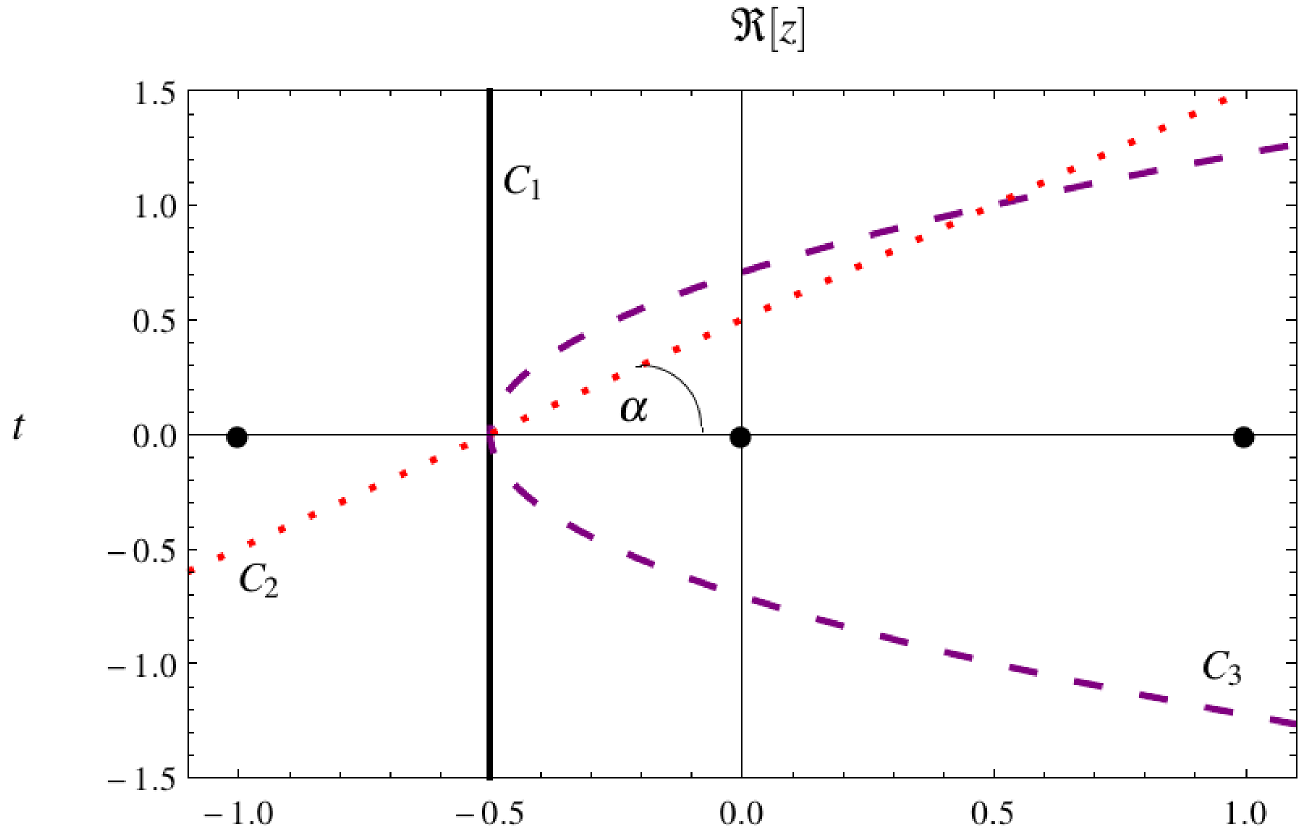


Figure 1: Integration contours chosen for the real part of the complex variable z defined in Eqs. (3.3),(3.5) and Eqs. (3.6)-(3.8). For C_2 $\alpha = \arctan(\frac{1}{\theta})$. Deformation from C_1 to C_2 or C_3 does not cross poles (black dots).

Specify to the kinematical value $s = 2$ (Minkowskian).
Get:

$$V_{-1}(2)|_{\text{analyt.}} = \frac{\pi}{4} = 0.785398163397448309 62.$$

$$\begin{aligned} V_{-1}(2)|_{C_1} &= 4.457455498513997 7188 + 4.513981236464512 2275\, i \\ V_{-1}(2)|_{C_2} &= 0.7853981633859819 - 5.420140575251864 \cdot 10^{-15}\, i \\ V_{-1}(2)|_{C_3} &= 0.7853981632958756 + 2.435551760271437 \cdot 10^{-15}\, i. \end{aligned}$$

Contour shifts

As an illustrative example of the efficiency of shifts, let us take the two-dimensional integrand

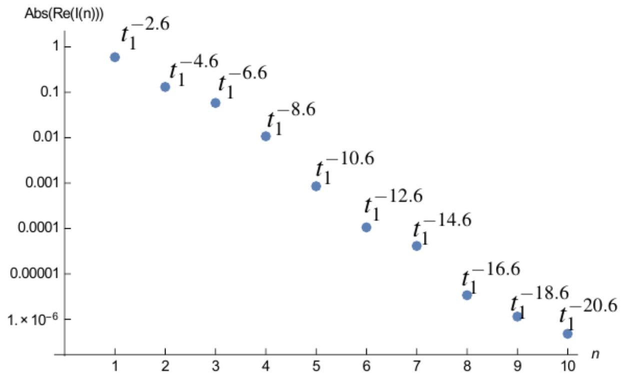
$$J(z_1,z_2) = \frac{2(-\frac{s}{M_Z^2})^{-z_1-z_2}\Gamma[-1-z_1-2z_2]\Gamma[-z_1-z_2]\Gamma[-z_2]\Gamma[1+z_2]^3\Gamma[1+z_1+z_2]}{s^2\Gamma[1-z_1]} \qquad (3.13)$$

The shift of contours ...

works well. To see this, we shift z_2 variable, $z_2 = z_{20} + n$. The integral is now a discrete function of the number of shifts n:

$$I^{C_1}(s,M_Z,n) = \int^{+\infty}\int^{+\infty}(i)^2 J(z_{10}+it_1, z_{20}+n+it_2)\mathrm{d}t_1\mathrm{d}t_2. \qquad (3.14)$$

Figure 2 shows that after shifting the integration contour to the right the main contribution comes from the crossed, simpler residues, while the contour integral contributes less and less - thus



rising the net accuracy.

Figure 2: Module of real part of the integral $I^{C_1}(s,M_Z,n)$ as a function of n.

A last example, where sector decomposition is not competative.
Analytical result: Bonciani et al. (2003) [34]

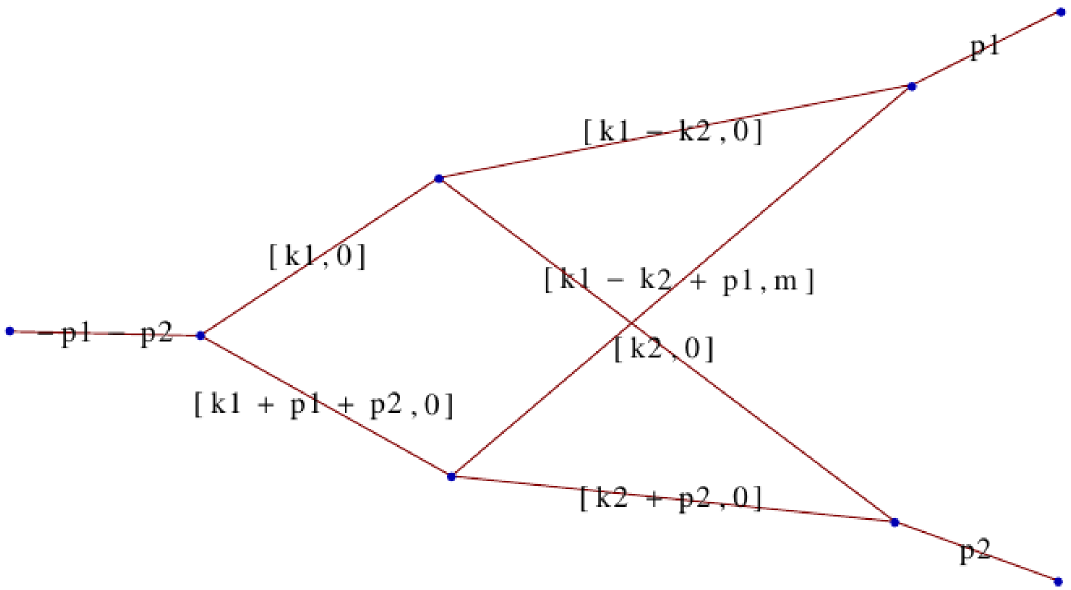


Figure 6: Non-planar vertex with one massive crossed line. Figure generated by `PlanarityTest` [19, 18].

Finally, we give as another example the constant part of the 3-dimensional integrand Eq. (3.29) drawn in Fig. 6

$$(-s)^{-2\varepsilon-z_2-2}\left(m^2\right)^{z_2}\Gamma[-\varepsilon]\Gamma[-z_1]\Gamma[-z_2]\Gamma[-z_3]\Gamma^2[z_3+1]\Gamma[-\varepsilon-z_1]\Gamma[-\varepsilon-z_2]\Gamma[z_1+z_3+1]$$
$$\times\frac{\Gamma[-2\varepsilon-z_1-z_3-1]\Gamma[-2\varepsilon-z_2-z_3-1]\Gamma[-2\varepsilon-z_1-z_2-z_3-1]\Gamma[2\varepsilon+z_1+z_2+z_3+2]}{\Gamma[-2\varepsilon-z_1]\Gamma[-3\varepsilon-z_2]\Gamma[-2\varepsilon-z_2]\Gamma[-2\varepsilon-z_1-z_2]}, \tag{3.29}$$

which shows how powerful `MBnumerics.m` can be. In this case, results obtained with different available methods and programs in the Euclidean region are the following, $-(p_1+p_2)^2 = m^2 = 1$:

$$\begin{array}{ll} \text{Analytical}: & -0.4966198306057021 \\ \text{MB(Vegas)}: & -0.4969417442183914 \\ \text{MB(Cuhre)}: & -0.4966198313219404 \\ \text{FIESTA}: & -0.4966184488196595 \\ \text{SecDec}: & -0.4966192150541896 \end{array}$$

For the Minkowskian region, $(p_1+p_2)^2 = m^2 = 1 + i\varepsilon$, constant part:

$$\begin{array}{lc} \text{Analytical}: & -0.778599608979684 - 4.123512593396311 \cdot i \\ \text{MBnumerics}: & -0.778599608324769 - 4.123512600516016 \cdot i \\ \text{MB(Vegas)}: & \text{big error} \\ \text{MB(Cuhre)}: & \text{NaN} \\ \text{FIESTA}: & \text{big error} \\ \text{SecDec}: & \text{big error} \end{array}$$

The role of dimensionality – as example the massive one-loop box integral $\rightarrow$ Phan , TR et al. $\rightarrow$ Usovitsch, MB1loop

From: [29, 36, 30, 37]

$$F_n(x) = -(\sum_i x_i)\, J + Q^2 = \frac{1}{2}\sum_{i,j} x_i Y_{ij} x_j - i\epsilon,$$

The Y_{ij} are elements of the Cayley matrix, introduced for a systematic study of one-loop n-Feynman integrals e.g. in [12]

$$Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2.$$

There are $N_n = \frac{1}{2}n(n+1)$ different Y_{ij} for n-point functions: $N_3 = 6, N_4 = 10, N_5 = 15$.

The recursion relation for 1-loop n-point functions

$$J_n(d,\{q_i,m_i^2\}) = \frac{-1}{2\pi i}\int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(\frac{d-n+1}{2}+s)\Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} R_n^{-s} \times \sum_{k=1}^{n}\left(\frac{1}{r_n}\frac{\partial r_n}{\partial m_k^2}\right) \mathbf{k}^- J_n(d+2s;\{q_i,m_i^2\}).$$

The cases $G_n = 0$ and $\lambda_n = r_n = 0$ prevent the use of the Mellin-Barnes transformation.
$\rightarrow$ Perform reductions to simpler functions a la [8].

1-point function, or tadpole

$$J_1(d;m^2) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2 - m^2 + i\epsilon} = -\frac{\Gamma(1-d/2)}{(m^2-i\epsilon)^{1-d/2}}.$$

$$J_4(12-2\epsilon,1,5,1,1) \to I^{[d+]^4}_{4,2222} = D_{1111}$$

x	value for $4! \times \; J_4(12-2\epsilon,1,5,1,1)$	
0	$(2.05969289730 + 1.55594910118i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
0	$(2.05969289730 + 1.55594910118i)10^{-10}$	MBOneLoop + Kira + MBnumerics
10^{-8}	$(2.05969289342 + 1.55594909187i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
10^{-8}	$(2.05969289363 + 1.55594909187i)10^{-10}$	MBOneLoop + Kira + MBnumerics
10^{-4}	$(2.05965609497 + 1.55585605343i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
10^{-4}	$(2.05965609489 + 1.55585605343i)10^{-10}$	MBOneLoop + Kira + MBnumerics

Table 4: *The Feynman integral $J_4(12-2\epsilon,1,5,1,1)$ as defined in (47) compared to numbe [5]. The $I^{[d+]^4}_{4,2222}$ is the scalar integral where propagator 2 has index $\nu_2 = 1 + (1+1+1$ 5, the others have index 1. The integral corresponds to D_{1111} in notations of LoopToo For $x = 0$, the Gram determinant vanishes. We see an agreement of about 10 to 11 r digits. The deviations of the two calculations seem to stem from a limited accuracy of the approximations used in [5].*

Embedding of 2-loop vertices into the real cross section calculations

The calculation of n-loop vertices is only part of the job for the determination of Z resonance electro-weak pseudo-observables – EWPOs.

The total cross section and asymmetries are defined generically through

$$\sigma_A^{real}(s) = \int \frac{ds'}{s} \rho_{tot}(s'/s)\, \sigma_A^{(0)}(s'), \quad A = tot, LR, pol, LRpol,$$

$$\sigma_A^{real}(s) = \int \frac{ds'}{s} \rho_{FB}(s'/s)\, \sigma_A^{(0)}(s'), \quad A = FB, LRFB, polFB, LRpolFB.$$

$$A_{FB}^{\text{meas}} = \frac{3}{4} A_e^{\text{th}} A_f^{\text{th}}$$

$$A_f^{\text{th}} = 2 \frac{\frac{v_f^2}{a_f^2}}{1 + \frac{v_f^2}{a_f^2}},$$

$$\frac{v_f}{a_f} = 1 - 4|Q_f| \sin^2\theta_{\text{eff}}^f \equiv 1 - 4|Q_f| \Re e(\kappa_f)\ \sin^2\theta_W$$

$$\mathcal{M}_\gamma^{(0)}(e^-e^+ \to f^-f^+) = \frac{4\pi i \alpha_{em}(s)}{s} Q_e Q_f \, \gamma_\alpha \otimes \gamma^\alpha,$$

$$\mathcal{M}_Z^{(0)}(e^-e^+ \to f^-f^+) = 4ie^2 \frac{\chi_Z(s)}{s} \Big[M_{vv}^{ef} \, \gamma_\alpha \otimes \gamma^\alpha - M_{av}^{ef} \, \gamma_\alpha \gamma_5 \otimes \gamma^\alpha$$
$$- M_{va}^{ef} \, \gamma_\alpha \times \gamma^\alpha \gamma_5 + M_{aa}^{ef} \, \gamma_\alpha \gamma_5 \otimes \gamma^\alpha \gamma_5 \Big].$$

$$M_{aa}^{ef} = I_e I_f \, \rho_Z = \pm \tfrac{1}{4} \rho_Z a_e^{ZF} a_v^{ZF},$$

$$\frac{M_{av}^{ef}}{M_{aa}^{ef}} \equiv 1 - 4|Q_f| \kappa_f \sin^2\theta_W = \frac{v_f^{ZF}}{a_f^{ZF}},$$

$$\frac{M_{va}^{ef}}{M_{aa}^{ef}} \equiv 1 - 4|Q_e| \kappa_e \sin^2\theta_W = \frac{v_e^{ZF}}{a_e^{ZF}},$$

$$\frac{M_{vv}^{ef}}{M_{aa}^{ef}} \equiv = 1 - 4(|Q_e|\kappa_e + |Q_f|\kappa_f) \sin^2\theta_W + 16|Q_e Q_f|^2 \sin^4\theta_W \kappa_{ef} = \frac{v_{ef}^{ZF}}{a_e^{ZF} a_f^{ZF}}$$

$$
\begin{aligned}
\mathcal{M}_{1,Z} &= \mathcal{M}_Z(e_L^- e_R^+ \to f_L^- f_R^+) = C\left[M_{aa}^{ef} + M_{av}^{ef} + M_{va}^{ef} + M_{vv}^{ef}\right]\frac{U}{s-s_0}, \\
\mathcal{M}_{2,Z} &= \mathcal{M}_Z(e_L^- e_R^+ \to f_R^- f_L^+) = C\left[-M_{aa}^{ef} + M_{av}^{ef} - M_{va}^{ef} + M_{vv}^{ef}\right]\frac{T}{s-s_0}, \\
\mathcal{M}_{3,Z} &= \mathcal{M}_Z(e_R^- e_L^+ \to f_L^- f_R^+) = C\left[-M_{aa}^{ef} - M_{av}^{ef} + M_{va}^{ef} + M_{vv}^{ef}\right]\frac{T}{s-s_0}, \\
\mathcal{M}_{4,Z} &= \mathcal{M}_Z(e_R^- e_L^+ \to f_R^- f_L^+) = C\left[M_{aa}^{ef} - M_{av}^{ef} - M_{va}^{ef} + M_{vv}^{ef}\right]\frac{U}{s-s_0}, \\
C &= 2\frac{G_F}{\sqrt{2}}\,\frac{\bar{M}_Z^2}{1+i\bar{\Gamma}_Z/\bar{M}_Z}.
\end{aligned}
$$

$$
\mathcal{M}_i(e^+e^- \to f\bar{f}) = \frac{s(1\pm\cos\theta)}{2}\left(Q_e Q_f \frac{4\pi\alpha_{em}(s)}{s} + \frac{R_f^{(i)}}{s-s_0} + \sum_{n=0}^{\infty} B_{f,n}^{(i)}(s-s_0)^n\right)
$$

$$
\begin{aligned}
A_{T,FB}^{(0)}(s) &= \frac{\sigma_{T,FB}^{(0)}(s)}{\sigma_T^{(0)}(s)} \\
&= \frac{\left[\int_0^{+1} - \int_{-1}^{0}\right] d\cos\theta \left(|\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + |\mathcal{M}_3|^2 + |\mathcal{M}_4|^2\right)}{\int_{-1}^{+1} d\cos\theta \left(|\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + |\mathcal{M}_3|^2 + |\mathcal{M}_4|^2\right)} \\
&= \frac{3}{4} \frac{2\,\Re e\{M_{av}^{ef}(M_{va}^{ef})^* + M_{aa}^{ef}(M_{vv}^{ef})^*\}}{|M_{aa}^{ef}|^2 + |M_{va}^{ef}|^2 + |M_{av}^{ef}|^2 + |M_{vv}^{ef}|^2} \\
&= \frac{3}{4} \frac{2\Re e\{a_e^{ZF} v_e^{ZF*} a_f^{ZF*} v_f^{ZF} + a_e^{ZF} a_f^{ZF} v_{ef}^{ZF*}\}}{|a_e^{ZF} a_f^{ZF}|^2 + |v_e^{ZF} a_f^{ZF}|^2 + |a_e^{ZF} v_f^{ZF}|^2 + |v_{ef}^{ZF}|^2} \\
&= \frac{3}{4} \frac{(a_e v_e^* + v_e a_e^*)(a_f v_f^* + v_f a_f^*) + \Delta_{FB}}{(|a_e|^2 + |v_e|^2)(|a_f|^2 + |v_f|^2) + \Delta_T}
\end{aligned}
$$

$$
\begin{aligned}
R_{vv} &= v_e^{(0)} R_{ZZ} v_f^{(0)} + \left[v_e^{(1)}(M_Z^2)\, v_f^{(0)} + v_e^{(0)}\, v_f^{(1)}(M_Z^2)\right]\left[1 + {\Sigma_{ZZ}^{(1)}}'(M_Z^2)\right] \\
&+ v_e^{(2)}(M_Z^2)\, v_f^{(0)} + v_e^{(0)}\, v_f^{(2)}(M_Z^2) + v_e^{(1)}(M_Z^2)\, v_f^{(1)}(M_Z^2) \\
&- iM_Z\Gamma_Z \left[{v_e^{(1)}}'(M_Z^2)\, v_f^{(0)} + v_e^{(0)}\, {v_f^{(1)}}'(M_Z^2)\right], \\
R_{ZZ} &= 1 - {\Sigma_{ZZ}^{(1)}}'(M_Z^2) \\
&- {\Sigma_{ZZ}^{(2)}}'(M_Z^2) + \left({\Sigma_{ZZ}^{(1)}}'(M_Z^2)\right)^2 + iM_Z\Gamma_Z\, {\Sigma_{ZZ}^{(1)}}''(M_Z^2) \\
&- \frac{1}{M_Z^4}\left(\Sigma_{\gamma Z}^{(1)}(M_Z^2)\right)^2 + \frac{2}{M_Z^2}\, \Sigma_{\gamma Z}^{(1)}(M_Z^2)\, {\Sigma_{\gamma Z}^{(1)}}'(M_Z^2).
\end{aligned}
$$

The so-called ZFITTER approach is a realistic option for a precision analysis of the Z-resonance measurements oat the FCC-ee Tera-Z

But with many checks and refinements

And combined with an S-matrix inspired parameterization in terms of Laurent series as in the Fortran package **SMATASY** [38, 39, 40, 41]

And with new software
See:
ZFITTER: [42, 43, 44, 45] and [46, 37]

New studies on the future role of EWPOs:
In several contributions by Dubovyk/Freitas/Gluza/Jadach/Riemann/Usovitsch in the report on the mini workshop at CERN, January 2018 [3]

Summary and Outlook

Precision, Precision, Precision

- For the FCC-ee Tera-Z we need
  – Complete electroweak 2-loops and complete QCD 3-loops – with 4 safe digits
  – Leading electroweak 3-loops and leading QCD 4-loops – with 1 safe digit
- This will probably be based on numerical methods,
  SD, MB, DR (sector decomposition, Mellin-Barnes representation, dispersion relations)
  With crucial support by analytical approaches
- Time scale: 2045
  $\rightarrow$ We do not yet have the final algorithms or even software, but are optimistic
- Also needed:
  2-loop box diagrams
  Understanding of γ_5 at 2 and 3 loops
- **Our COST meeting at Katowice shall help to form a kind of collaboration of interested scientists!**

Some useful references

Dubovyk/Gluza/Freitas/Riemann/Usovitsch Electroweak 2-loop contributions to the Z vertex [47, 48, 49]

Riemann:April2018A-href [30]

G. t'Hooft and M. Veltman [9]

G. Passarino [50]

meromorphic functions, depending on multiple hypergeometric functions of the type ${}_2F_1, F_1, F_S$ [51, 28]

Phan, Bluemlein, Riemann:2017rbi APP 2017 [29]

The Cayley matrix $\lambda_{12\ldots n}$ was introduced in Melrose:1965kb [52]

Usovitsch:April2018A-href [53]

Let us remind the simple one-loop massive QED vertex for which no trivial MB method exists when the kinematics is Minkowskian, a problem raised in Czakon:2005rk [13] and solved in Dubovyk:2016ocz [35].

Davydychev:1991va [54]

Tkachov:1981wb, Chetyrkin:1981qh [55, 56]

Usyukina:1975yg [21]

MB-suite AMBRE/MB/mbtools/MBnumerics/CUBA

Gluza:2007rt,Gluza:2010rn,Dubovyk:2016ocz [11, 1, 35],

Czakon:2005rk [13],

mbtools-href-m [57],
Usovitsch:201511-href [58],
Hahn:2004fe [16].
Smirnov:1999gc [23]
Tausk:1999vh [24]
Usovitsch:January2018A-hrefB [36]
secdec
fiesta