Scalar 1-loop Feynman integrals in arbitrary space-time dimension D

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https://indico.cern.ch/event/766859/timetable/

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See also:

K. H. Phan and T. Riemann
"Scalar 1-loop Feynman integrals as meromorphic functions in space-time dimension d,"
arXiv:1812.10975 [hep-ph] [1], to be submitted.

T. Riemann, K. H. Phan and J. Blümlein, talk at Podlesice2017, http://indico.if.us.edu.pl/event/4/contribution/32/material/slides/0.pdf.

K. H. Phan, J. Blümlein, and T. Riemann, Acta Phys. Polon. B48 (2017) 2313.

T. Riemann, K. H. Phan and J. Blümlein, talk at LL2018, unpublished. https://indico.desy.de/indico/event/16613/session/12/contribution/24/material/slides/0.pdf.

and

J. Usovitsch and T. Riemann, New approach to Mellin-Barnes integrals for massive one-loop Feynman integrals, section E.6 in Blondel et al. [2], and references therein

J. Usovitsch, I. Dubovyk, and T. Riemann, The MBnumerics project, section E.2 in Blondel et al., [2], and references therein

J. Usovitsch, Mathematica/Fortran packages MBnumerics and MBOneLoop

J. Usovitsch, PhD thesis, Humboldt-Universität zu Berlin (29 May 2018), PhD_Usovitsch_HUB_2018, http://dx.doi.org/10.18452/19484.

A. Blondel, J. Gluza, S. Jadach, P. Janot, T. Riemann (eds.), Standard Model Theory for the FCC-ee: The Tera-Z, subm. as CERN Yellow Report, 1809.01830.

Arbitrary space-time d $\longrightarrow$ Feynman integrals as meromorphic functions

In dimensional regularization, Feynman integrals depend on $d = 4 - 2\varepsilon$ and may be expanded in Laurent series:

$$I = \sum_{k=-n}^{\infty} \frac{a_k}{\varepsilon^k} \tag{1}$$

In fact, the Feynman integrals are meromorphic functions of d.
This means they are analytical in d everywhere with exclusion of isolated singular points d_s, where they behave not worse than

$$\frac{A_s}{(d - d_s)^{N_s}} \tag{2}$$

This raises the question:
Can we determine the complete d-dependence of a Feynman integral?

Similar problem, also of practical relevance: The Z boson at the FCCee
The scattering amplitude of the Z boson resonance as a function of the "energy" (or: s) is a resonance curve with one simple pole, defining the mass and the width of the Z boson:

$$A = \frac{R}{s - M_Z^2 + iM_Z\Gamma_Z} + \sum_{i=0}^{\infty} a_i (s - M_Z^2 + iM_Z\Gamma_Z)^i \tag{3}$$

Why one-loop Feynman integrals in $D = 4 + 2n - 2\epsilon$ dimensions?

Basics

The seminal papers on 1-loop Feynman integrals:
't Hooft, Veltman, Nov. 1978 [3]: "Scalar oneloop integrals"
Passarino, Veltman, Feb. 1978 [4]: "One Loop Corrections for e^+e^- Annihilation into $\mu^+\mu^-$ in the Weinberg Model"

1. Interest in 1-loop integrals from n-point tensor reductions

For many-particle calculations, there appear inverse Gram determinants from tensor reductions in the answers.

These $1/G(p_i)$ may diverge, because Gram dets can exactly vanish: $G(p_i) \equiv 0$.

One may perform tensor reductions so that no inverse Grams appear, but one has to buy 1-loop integrals in higher dimensions, $D = 4 + 2n - 2\epsilon$. See [5, 6].
Higher propagator exponents ("indices") may be avoided:
For two- to seven-point tensor functions this has been worked out in Riemann et al., Radcor2013 [6, 7].

From slides of T. Riemann at Radcor2013:

https://conference.ippp.dur.ac.uk/event/341/session/8/contribution/56/material/slides/0.pdf

As an example, we reproduce the 4-point part of $I_{4,ijkl}^{[d+]^4}$:

$$
\begin{aligned}
n_{ijkl} I_{4,ijkl}^{[d+]^4} &= \frac{\binom{0}{i}}{\binom{0}{0}} \frac{\binom{0}{j}}{\binom{0}{0}} \frac{\binom{0}{k}}{\binom{0}{0}} \frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+]^4} \\
&+ \frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0j}{0k}\binom{0}{i}\binom{0}{l} + \binom{0i}{0l}\binom{0}{j}\binom{0}{k} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3} \\
&\times d(d+1) I_4^{[d+]^3} \\
&+ \frac{\binom{0i}{0l}\binom{0j}{0k} + \binom{0j}{0l}\binom{0i}{0k} + \binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+]^2} + \cdots
\end{aligned}
\tag{12}
$$

Reducing the dimension to $d = 4 - 2\varepsilon$ a la Tarasov would introduce inverse Gram determinants.

2. Interest in 1-loop integrals from multi-loops

Higher-order loop calculations need h.o. contributions from ϵ-expansions of 1-loops:
$1/(d-4) = -1/(2\epsilon)$ and $\Gamma(\epsilon) = a_1/\epsilon + a_0 + a_1\epsilon + \cdots$

A Seminal paper was on ϵ-terms of 1-loop functions:
Nierste, Müller, Böhm, 1992 [8]: "Two loop relevant parts of D-dimensional massive scalar one loop integrals"

3. Interest in 1-loop integrals from Mellin-Barnes integrals, see AMBRE loop-by-loop approach

One-loop integrals with variable indices are also needed in the context of the loop-by-loop Mellin-Barnes approach to multi-loop integrals of the Mathematica package AMBRE [9, 10, 11].
Many scales lead, with AMBRE, to multi-dimensional Mellin-Barnes integrals.

Conclusion → 1-loop integrals in d dimensions are of interest

A general solution in d dimensions was derived in another 2 seminal papers:
Tarasov, 2000 [12] – massive vertex

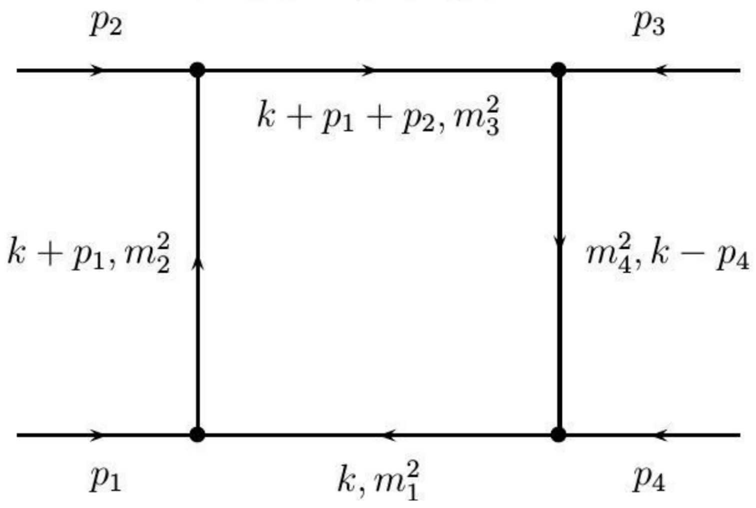
Discovered the need of
$_2F_1$ **Gauss** hypergeometric function for self-energy
F_1 **Appell** function for vertices
F_S **Lauricella-Saran** function for box integrals

Fleischer, Jegerlehner, Tarasov, 2003 [13], massive box:
"A New hypergeometric representation of one loop scalar integrals in d dimensions"

I was wondering if the results of Fleischer/Jegerlehner/Tarasov (2003) are useful for deriving numerical black-box software applications?

There was no explicit numerics in the papers [12, 13].
As mentioned, we came back to the approach in 2013 with Jochem Fleischer.
Best would be a complete re-derivation.

Phan, Blümlein, Riemann, 2015: Restart with new, independent approach



$$J_N \equiv J_N(d; \{p_i p_j\}, \{m_i^2\}) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_N^{\nu_N}} \tag{4}$$

with

$$D_i = \frac{1}{(k+q_i)^2 - m_i^2 + i\epsilon}. \tag{5}$$

$$\nu_i = 1, \quad \sum_{i=1}^{n} p_i = 0 \tag{6}$$

$$J_n = (-1)^n \Gamma\left(n - d/2\right) \int_0^1 \prod_{j=1}^{n} dx_j \delta\left(1 - \sum_{i=1}^{n} x_i\right) \frac{1}{F_n(x)^{n-d/2}} \tag{7}$$

Here, the F-function is the second Symanzik polynomial.

$$F_n(x) = -(\sum_i x_i)\, J + Q^2 = \frac{1}{2} \sum_{i,j} x_i Y_{ij} x_j - i\epsilon, \tag{8}$$

Use Mellin-Barnes integrals to split the sum in $F_n(x)$ into a product, getting nested MB-integrals to be calculated.

The Y_{ij} are elements of the Cayley matrix, introduced for a systematic study of one-loop n-point Feynman integrals e.g. in [14]

$$Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2. \tag{9}$$

There are $N_n = \frac{1}{2} n(n+1)$ different Y_{ij} for n-point functions.
This leads to $N_n - 1$ dimensional Mellin-Barnes integrals.
$N_3 - 1 = 5$ **– massive vertex**
$N_4 - 1 = 9$ **– massive box**
$N_5 - 1 = 14$ **– massive pentagon**

Mellin-Barnes representation

$$\frac{1}{(1+z)^\lambda} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds\, \frac{\Gamma(-s)\,\Gamma(\lambda+s)}{\Gamma(\lambda)}\, z^s = {}_2F_1\left[\begin{array}{c}\lambda, b\,; \\ b\,;\end{array} -z\right]. \qquad (10)$$

Eqn. (10) is valid if $|\mathrm{Arg}(z)| < \pi$.
The integration contour has to be chosen such that the poles of $\Gamma(-s)$ and $\Gamma(\lambda+s)$ are well-separated. The right hand side of (10) is identified as Gauss' hypergeometric function. For more details see [15]).

F-function and Gram and Cayley determinants

Gram and Cayley det's were introduced by Melrose (1965) [14]. The Cayley determinant $\lambda_{12\ldots N}$ is composed of the
$Y_{ij} = m_i^2 + m_j^2 - (q_i - q_j)^2$ introduced in (9), and its determinant is:

$$\text{Cayley determinant :} \qquad \lambda_n \equiv \lambda_{12\ldots n} \quad = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix} . \tag{11}$$

We also define the $(n-1)\times(n-1)$ dimensional Gram determinant $g_n \equiv g_{12\cdots n}$,

$$G_n \equiv G_{12\cdots n} = - \begin{vmatrix} (q_1-q_n)^2 & (q_1-q_n)(q_2-q_n) & \ldots & (q_1-q_n)(q_{n-1}-q_n) \\ (q_1-q_n)(q_2-q_n) & (q_2-q_n)^2 & \ldots & (q_2-q_n)(q_{n-1}-q_n) \\ \vdots & \vdots & \ddots & \vdots \\ (q_1-q_n)(q_{n-1}-q_n) & (q_2-q_n)(q_{n-1}-q_n) & \ldots & (q_{n-1}-q_n)^2 \end{vmatrix} . \tag{12}$$

Both determinants are independent of a common shifting of the momenta q_i.
Further, the Gram det G_n is independent of the propagator masses.

Rewriting the F-function further, exploring the $\delta(1-\sum x_i)$ ...

The δ-function: The elimination of x_n, one of the x_i, creates linear terms in $F(x)$.

$$F_n(x) = (x-y)^T G_n (x-y) + R_n, \tag{13}$$

$$R_n = r_n - i\varepsilon = -\frac{\lambda_n}{g_n} - i\varepsilon. \tag{14}$$

The inhomogeneity $R_n = r_n - i\varepsilon$ carries the $i\varepsilon$-prescription.

The recursion relation for 1-loop n-point functions

$$J_n(d,\{q_i,m_i^2\}) = \frac{-1}{2\pi i}\int_{-i\infty}^{+i\infty} ds\, \frac{\Gamma(-s)\,\Gamma(\frac{d-n+1}{2}+s)\Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} R_n^{-s} \times \sum_{k=1}^{n}\left(\frac{1}{r_n}\frac{\partial r_n}{\partial m_k^2}\right)\mathbf{k}^- J_n(d+2s;\{q_i,m_i^2\}). \qquad (15)$$

The operator $\mathbf{k}^-$ ... ...will reduce an n-point Feynman integral J_n to an $(n-1)$-point integral J_{n-1} by shrinking the propagator $1/D_k$.

The cases $G_n = 0$ and $\lambda_n = r_n = 0$ prevent the use of the Mellin-Barnes transformation.

1-point function, or tadpole

$$J_1(d;m^2) = \int \frac{d^dk}{i\pi^{d/2}}\frac{1}{k^2-m^2+i\varepsilon} = -\frac{\Gamma(1-d/2)}{R_1^{1-d/2}} \qquad (16)$$

$$R_1 = m^2 - i\varepsilon \qquad (17)$$

Comments

- In Tarasov 2003 [13], an **infinite sum** was to be solved. Formulae for 2,3,4-point functions are given.
- Any 4-point integral e.g. is in our recursion a **3-fold Mellin-Barnes** integral. With AMBRE, we get for e.g. box integrals up to **9-fold** MB-integrals
- Euklidean and Minkoswkian integrals converge equally good. See J. Usovitsch's talk at LL2018 [16].

R [17].

Eqn. (19) of Tarasov et al., 2003 [13]:

by redefining the l independent 'boundary term' $\tilde{b}_n$. The final result for $I_n^{(d)}$ then reads

$$I_n^{(d)} = b_n(\varepsilon) - \sum_{k=1}^{n} \left(\frac{\partial_k \Delta_n}{2\Delta_n}\right) \sum_{r=0}^{\infty} \left(\frac{d-n+1}{2}\right)_r \left(\frac{G_{n-1}}{\Delta_n}\right)^r \mathbf{k}^- I_n^{(d+2r)}. \qquad (19)$$

As we will show in the next sections, b_n can be determined from the asymptotic behavior of $I_n^{(d)}$ for $d \to \infty$ or by setting up a differential equation for it. This term depends on the kinematic domain.

The 2-point function

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The massive vertex function

$J_3 = J_{123} + J_{231} + J_{312}$ with $R_3 = R_{123}, R_2 = R_{12}$ etc.

$$J_{123} = \Gamma\left(2-\frac{d}{2}\right)\frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{r_2}\frac{r_2}{2\sqrt{1-m_1^2/r_2}}$$

$$\left[-R_2^{d/2-2}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)}\,{}_2F_1\left[\begin{matrix}\frac{d-2}{2},1; \\ \frac{d-1}{2};\end{matrix}\ \frac{R_2}{R_3}\right] + R_3^{d/2-2}\,{}_2F_1\left[\begin{matrix}1,1; \\ 3/2;\end{matrix}\ \frac{R_2}{R_3}\right]\right]$$

$$+\Gamma\left(2-\frac{d}{2}\right)\frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{r_2}\frac{m_1^2}{4\sqrt{1-m_1^2/r_2}}$$

$$\left[+\frac{2(m_1^2)^{d/2-2}}{d-2}F_1\left(\frac{d-2}{2};1,\frac{1}{2};\frac{d}{2};\frac{m_1^2}{R_3},\frac{m_1^2}{R_2}\right) - R_3^{d/2-2}F_1\left(1;1,\frac{1}{2};2;\frac{m_1^2}{R_3},\frac{m_1^2}{R_2}\right)\right]$$

$$+\ (m_1^2 \leftrightarrow m_2^2)$$

For $d \to 4$, both the $[\ldots]$ approach zero.

So the J_3 is finite in this limit, as it should be for a massive 3-point function.

Tarasov (2003) [13], Eqns. (73) and (75)

There are kinematic conditions on internal momenta $q_{ij}^2 = (q_i - q_j)^2$ to be respected; the b_3-term of Tarasov becomes:

$$J_3(b_3) = \theta(-G_3) \times \theta(q_{ij}^2) \times \theta(\frac{m_i^2}{r_3} - 1) \times \frac{\Gamma(2-d/2)}{\lambda_3} \left(2^{3/2}\,\pi\,\sqrt{-G_3}\;R_3{}^{d/2-1}\right) \quad (18)$$

Otherwise:

$$J_3(b_3) = b_3 = 0. \quad (19)$$

Numerics for 3-point functions, table 2

$[p_i^2]$, $[m_i^2]$	[-100, +200, -300], [10, 20, 30]	
G_{123} λ_3 m_i^2/r_3 m_i^2/r_{12} m_i^2/r_{23} m_i^2/r_{31} $\sum J$-terms	480000 -19300000 0.248705, 0.497409, 0.746114 0.248447, 0.496894, 0.745342 -0.39801, -0.79602, -1.19403 0.104895, 0.20979, 0.314685 (-0.012307377 - 0.056679689 I)	 + (+ 0.012825498 I)/eps
$\sum b_3$-terms J_3(TR)	(+ 0.047378343 I) (-0.012307377 - 0.009301346 I)	- (+ 0.012825498 I)/eps
b_3-term $b_3 + \sum J$-terms	(+ 0.047378343 I) (-0.012307377 - 0.009301346 I)	- (+ 0.012825498 I)/eps
J_3(OT)	$\sum J$-terms, b_3-term$\to$0, **gets wrong!**	
MB suite		
(-1)*fiesta3	-(0.012307 + 0.009301 I)	+ (8*10-6 + 0.00001 I) pm4)
LoopTools/FF, ϵ^0	-0.0123073773677820630 - 0.0093013461700863289 i	

Table 1: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. Red input quantities suggest that, according to eq. (73) in Tarasov2003 [13], one has to set $b_3 = 0$. Further, we have set in the numerics for eq. (75) of Tarasov2003 [13] that Sqrt[-g123 + I*epsil], what looks counter-intuitive for a “momentum”-like function.

Both results agree if we *do not* set Tarasov's $b_3 \to 0$.

The massive box function J_4

$J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}$ is, with $R_4 = R_{1234}, R_3 = R_{123}, R_2 = R_{12}$ etc.:

$$
\begin{aligned}
J_{1234} \quad = \quad & \Gamma\left(2-\frac{d}{2}\right)\frac{\partial_4 r_4}{r_4}\Bigg\{ \\
& \left[\frac{b_{123}}{2}\left(-R_3^{d/2-2}\,{}_2F_1\left[\begin{matrix}\frac{d-3}{2},1; & \frac{R_2}{R_3}\\ \frac{d}{2}-1; & \end{matrix}\right]+R_4^{d/2-2}\sqrt{\pi}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}\,{}_2F_1(d\to 4)\right)\right] \\
& +\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}\frac{\sqrt{\pi}}{4}\frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{\sqrt{1-m_1^2/R_2}}\,{}_2F_1\left[\begin{matrix}1/2,1; & \frac{R_2}{R_3}\\ 1; & \end{matrix}\right] \\
& \left[+\frac{R_2^{d/2-2}}{d-3}F_1\left(\frac{d-3}{2};1,\frac{1}{2};\frac{d-1}{2};\frac{R_2}{R_4},\frac{R_2}{R_3}\right)-R_4^{d/2-2}F_1\left(d\to 4\right)\right] \\
& \frac{m_1^2}{8}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}\frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{r_2}\frac{r_3}{r_3-m_1^2}\frac{r_2}{r_2-m_1^2} \\
& \left[-(m_1^2)^{d/2-2}\frac{\Gamma\left(\frac{d}{2}-3/2\right)}{\Gamma\left(\frac{d}{2}\right)}F_S(d/2-3/2,1,1,1,1,d/2,d/2,d/2,d/2,\frac{m_1^2}{R_4},\frac{m_1^2}{m_1^2-R_3},\frac{m_1^2}{m_1^2-R_2})\right. \\
& \left.+R_4^{d/2-2}\sqrt{\pi}F_S(d\to 4)\right] \quad + \quad (m_1^2\leftrightarrow m_2^2)\Bigg\}
\end{aligned}
\tag{20}
$$

For $d\to 4$,all three [...] approach zero.
So that the massive J_4 gets finite then: OK.

The cases of vanishing Cayley determinant $\lambda_n = 0$ and of vanishing Gram determinant $G_n = 0$

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From Phan/Riemann 2018: The massive 1-loop box

Tab. 1: Comparison of the box integral J_4 defined in [17] with the LoopTools function D0 $(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2)$ **[19, 20] at $m_2^2 = m_3^2 = m_4^2 = 0$. Further numerical references are the packages K.H.P_D0 (PHK, unpublished) and MBOneLoop [18]. External invariants:** $(p_1^2 = \pm 1, p_2^2 = \pm 5, p_3^2 = \pm 2, p_4^2 = \pm 7, s = \pm 20, t = \pm 1)$.

$(p_1^2, p_2^2, p_3^2, p_4^2, s, t)$	4-point integral
$(-,-,-,-,-,-)$	$d = 4, \quad m_1^2 = 100$
J_4	0.00917867
LoopTools	0.0091786707
MBOneLoop	0.0091786707
$(+,+,+,+,+,+)$	$d = 4, \quad m_1^2 = 100$
J_4	$-0.0115927 - 0.00040603\, i$
LoopTools	$-0.0115917 - 0.00040602\, i$
MBOneLoop	$-0.0115917369 - 0.0004060243\, i$
$(-,-,-,-,-,-)$	$d = 5, \quad m_1^2 = 100$
J_4	0.00926895
K.H.P_D0	0.00926888
MBOneLoop	0.0092689488
$(+,+,+,+,+,+)$	$d = 5, \quad m_1^2 = 100$
J_4	$-0.00272889 + 0.0126488\, i$
K.H.P_D0	(–)
MBOneLoop	$-0.0027284242 + 0.0126488134\, i$
$(-,-,-,-,-,-)$	$d = 5, \quad m_1^2 = 100 - 10\, i$
J_4	$0.00920065 + 0.000782308\, i$
K.H.P_D0	$0.0092006 + 0.000782301\, i$
MBOneLoop	$0.0092006481 + 0.0007823090\, i$
$(+,+,+,+,+,+)$	$d = 5, \quad m_1^2 = 100 - 10\, i$
J_4	$-0.00398725 + 0.012067\, i$
K.H.P_D0	$-0.00398723 + 0.012069\, i$
MBOneLoop	$-0.0039867702 + 0.0120670388\, i$

Gauss ${}_2F_1$ and Appell function F_1 and Saran function F_S

Numerical calculations of specific Gauss hypergeometric functions ${}_2F_1$, Appell functions F_1 (Eqn. (1) of [18]), and Lauricella-Saran functions F_S (Eqn. (2.9) of [19]) are needed for the scalar one-loop Feynman integrals:

$$ {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!\ (c)_k}\ x^k, \tag{21} $$

$$ F_1(a;b,b';c;y,z) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!\ n!\ (c)_{m+n}}\ y^m z^n, \tag{22} $$

$$ F_S(a_1,a_2,a_2;b_1,b_2,b_3;c,c,c;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_{n+p}(b_1)_m(b_2)_n(b_3)_p}{m!\ n!\ p!\ (c)_{m+n+p}}\ x^m y^n z^p. \tag{23} $$

The $(a)_k$ is the Pochhammer symbol.

Mellin-Barnes integrals for ${}_2F_1$ and F_1 and F_S

One approach to the numerics of F_1 and F_S may be based on Mellin-Barnes representations. For the Gauss function ${}_2F_1$ and the Appell function F_1, Mellin-Barnes representations are known. See Eqn. (1.6.1.6) in [20],

$$ {}_2F_1(a,b;c;z) = \frac{1}{2\pi i}\,\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{+i\infty} ds\;(-z)^s\,\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}, \qquad (24) $$

and Eqn. (10) in [18], which is a two-dimensional MB-integral:

$$ F_1(a;b,b';c;x,y) = \frac{1}{2\pi i}\,\frac{\Gamma(c)}{\Gamma(a)\Gamma(b')} \int_{-i\infty}^{+i\infty} dt\;(-y)^t\;{}_2F_1(a+t,b;c+t,x)\frac{\Gamma(a+t)\Gamma(b'+t)\Gamma(-t)}{\Gamma(c+t)}. \qquad (25) $$

For the Lauricella-Saran function F_S we derived the following, new, three-dimensional MB-integral:

$$ F_S(a_1,a_2,a_2;b_1,b_2,b_3;c,c,c;x,y,z) = \frac{1}{2\pi i}\,\frac{\Gamma(c)}{\Gamma(a_1)\Gamma(b_1)} \int_{-i\infty}^{+i\infty} dt(-x)^t\,\frac{\Gamma(a_1+t)\Gamma(b_1+t)\Gamma(-t)}{\Gamma(c+t)} \times\; F_1(a_2;b_2,b_3;c+t;y,z). \qquad (26) $$

Calculate F_1

We advocate the numerical mean value integration of the following integral representations:
A one-dimensional integral representation for F_1 [21] is quoted in Eqn. (9) of [18]:

$$F_1(a;b,b';c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 du \frac{u^{a-1}(1-u)^{c-a-1}}{(1-xu)^b(1-yu)^{b'}}. \tag{27}$$

We need three specific cases, taken at $d \geq 4$.

For vertices

$$F_1^v(d) \equiv F_1\left(\frac{d-2}{2};1,\frac{1}{2};\frac{d}{2};x_c,y_c\right) = \frac{1}{2}(d-2)\int_0^1 \frac{du\, u^{\frac{d}{2}-2}}{(1-x_c u)\sqrt{1-y_c u}}. \tag{28}$$

Integrability is violated at $u=0$ if not $\Re e(d) > 2$.

Calculate F_S

For the calculation of the 4-point Feynman integrals, one needs the Lauricella-Saran function F_S [19]. Saran defines F_S as three-fold sum (23), see Eqn. (2.9) in [19]. He derives a 3-fold integral representation in Eqn. (2.15) and a 2-fold integral in Eqn. (2.16). We will use the following quite useful representation, derived at p. 304 of [19]:

$$F_S(a_1, a_2, a_2; b_1, b_2, b_3; c, c, c, x, y, z) = \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(c-a_1)} \int_0^1 dt \frac{t^{c-a_1-1}(1-t)^{a_1-1}}{(1-x+tx)^{b_1}} F_1(a_2; b_2, b_3; c-a_1; ty, tz). \tag{29}$$

For box integrals

$$F_S^b(d) = F_S\left(\frac{d-3}{2}, 1, 1; 1, 1, \frac{1}{2}; \frac{d}{2}, \frac{d}{2}, \frac{d}{2}, x_c, y_c, z_c\right) = \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-3}{2})\Gamma(\frac{3}{2})} \int_0^1 dt \frac{\sqrt{t}(1-t)^{\frac{d-5}{2}}}{(1-x_c+x_c t)} F_1(1; 1, \frac{1}{2}; \frac{3}{2}; y_c t, z_c t) \tag{30}$$

Eqn. (30) is valid if $\Re e(d) > 3$.

From Phan/Riemann 2018: Appell function F_1

Tab. B.1: The Appell function F_1 of the massive vertex integrals as defined in (B.2). As a proof of principle, only the constant term of the expansion in $d = 4 - 2\varepsilon$ is shown, $F_1(1; 1, \frac{1}{2}; 2; x, y)$. Upper values: this calculation, (Appendix B.2), lower values: (B.7).

$x - i\varepsilon_x$	$y - i\varepsilon_y$	$F_1(1;1,\frac{1}{2};2;x,y)$	
$+11.1 - 10^{-12}i$	$+12.1 - 10^{-12}i$	-0.1750442480735	$-0.0542281294732\,i$
		$-0.17504424807351877884498289912$	$-0.054228129473304027882097641167\,i$
$+11.1 - 10^{-12}i$	$+12.1 + 10^{-12}i$	$+1.7108545293244$	$+0.0542281294732\,i$
		$+1.71085452932433557134838204175$	$+0.05422812947148217381589270924\,i$
$+11.1 + 10^{-12}i$	$+12.1 - 10^{-12}i$	$+1.7108545304114$	$-0.0542281294732\,i$
		$+1.71085452932433557134838204175$	$-0.05422812947148217381589270924\,i$
$+11.1 + 10^{-12}\,i$	$+12.1 + 10^{-12}\,i$	-0.1750442480735	$+0.0542281294733\,i$
		$-0.17504424807351877884498289912$	$+0.054228129473304027882097641167\,i$
$+12.1 - 10^{-15}\,i$	$+11.1 - 10^{-15}\,i$	-0.1700827166484	$-0.0518684846037\,i$
$+12.1 - 10^{-10}\,i$	$+11.1 - 10^{-15}\,i$	$-0.17008271664800058101165749279$	$-0.05186848460465674976556525621\,i$
$+12.1 - 10^{-15}\,i$	$+11.1 + 10^{-15}\,i$	-0.1700827166484	$-1.7544202909955\,i$
		$-0.17008271664844025647268817399$	$-1.75442029099557688735842562038\,i$
$+12.1 + 10^{-15}\,i$	$+11.1 - 10^{-15}\,i$	-0.1700827166484	$+1.7544202909955\,i$
		$-0.17008271664844025647268817399$	$+1.75442029099557688735842562038\,i$
$+12.1 + 10^{-15}\,i$	$+11.1 + 10^{-15}\,i$	-0.1700827166484	$+0.0518684846037\,i$
$+12.1 - 10^{-10}\,i$	$+11.1 - 10^{-15}\,i$	$-0.17008271664800058101165749279$	$+0.05186848460465674976556525621\,i$
$+11.1 - 10^{-15}\,i$	-12.1	-0.0533705146518	$-0.1957692111557\,i$
		$-0.05337051465189944473349401152$	$-0.195769211155733985388920833693\,i$
$+11.1 + 10^{-15}\,i$	-12.1	-0.0533705146518	$+0.1957692111557\,i$
		$-0.05337051465189944473349401152$	$+0.195769211155733985388920833693\,i$
-11.1	$+12.1 - 10^{-12}\,i$	$+0.1060864084662$	$-0.1447440700082i$
		$+0.10608640847651064287133527599$	$-0.144744070021333407167349619088\,i$
-11.1	$+12.1 + 10^{-12}\,i$	$+0.1060864084662$	$+0.1447440700082i$
		$+0.10608640847651064287133527599$	$+0.144744070021333407167349619088\,i$
-12.1	-11.1	$+0.122456767687224028$	
		$+0.12245676768722402506513395161$	

From Phan/Riemann 2018: Appell function F_1

Tab. B.2: The Appell function $F_1(1-\epsilon; 1, \frac{1}{2}; 2-\epsilon; x_c, y_c)$ **as defined in (B.2), needed for** $d = 4-2\varepsilon$ **at** $x_c = 11.1 - 10^{-12}\, i$, $y_c = 12.1 - 10^{-12}\, i$.

$F_1(1-\epsilon; 1, \frac{1}{2}; 2-\epsilon; x_c, y_c)$	
$+(-0.17504424807035$	$-0.05422812947328\, i)$
$+(-0.00861885585913$	$-0.39051763820462\, i)\epsilon$
$+(+0.37518853545319$	$-0.34047477405516\, i)\epsilon^2$
$+(+0.49765461883470$	$-0.00717399489427\, i)\epsilon^3$
$+(+0.32835724868237$	$+0.23005850008124\, i)\epsilon^4$
$+(+0.11199125312340$	$+0.25409725390712\, i)\epsilon^5$
$+(-0.00954795237038$	$+0.17050760870656\, i)\epsilon^6$
$+(-0.04217861994524$	$+0.08576862780838\, i)\epsilon^7$

Massive box as 3-fold Mellin-Barnes integral I

And finally we reproduce the box integral, dependent on d and the internal variables $\{d, q_1, m_1^2, \cdots q_4, m_4^2\}$ or, equivalently, on a set of external variables, e.g. $\{d, \{p_i^2\}, \{m_i^2\}, s, t\}$:

$$\begin{aligned} J_4(d; \{p_i^2\}, s, t, \{m_i^2\}) \quad = \quad & \left(\frac{-1}{4\pi i}\right)^4 \frac{1}{\Gamma(\frac{d-3}{2})} \sum_{k_1,k_2,k_3,k_4=1}^{4} D_{k_1k_2k_3k_4} \left(\frac{1}{r_4}\frac{\partial r_4}{\partial m_{k_4}^2}\right) \\ & \left(\frac{1}{r_{k_3k_2k_1}}\frac{\partial r_{k_3k_2k_1}}{\partial m_{k_3}^2}\right)\left(\frac{1}{r_{k_2k_1}}\frac{\partial r_{k_2k_1}}{\partial m_{k_2}^2}\right)(m_{k_1}^2)^{d/2-1} \\ & \int_{-i\infty}^{+i\infty} dz_4 \int_{-i\infty}^{+i\infty} dz_3 \int_{-i\infty}^{+i\infty} dz_2 \left(\frac{m_{k_1}^2}{R_4}\right)^{z_4}\left(\frac{m_{k_1}^2}{R_{k_3k_2k_1}}\right)^{z_3}\left(\frac{m_{k_1}^2}{R_{k_2k_1}}\right)^{z_2} \\ & \Gamma(-z_4)\Gamma(z_4+1)\frac{\Gamma(z_4+\frac{d-3}{2})}{\Gamma(z_4+\frac{d-2}{2})}\Gamma(-z_3)\Gamma(z_3+1)\frac{\Gamma(z_3+z_4+\frac{d-2}{2})}{\Gamma(z_3+z_4+\frac{d-1}{2})} \\ & \Gamma(z_2+z_3+z_4+\frac{d-1}{2})\Gamma(-z_2-z_3-z_4-\frac{d+2}{2})\Gamma(-z_2)\Gamma(z_2+1). \end{aligned} \tag{31}$$

The representation (31) can be treated by the Mathematica packages MB and MBnumerics of the MBsuite, replacing AMBRE by a derivative of MBnumerics: **MBOneLoop** [22].

Example: A massive 4-point function with vanishing Gram determinant

Left out here, but one table.
See [22].

$J_4(12-2\epsilon,1,5,1,1) \to I_{4,2222}^{[d+]^4} = D_{1111}$

x	value for $4! \times J_4(12-2\epsilon,1,5,1,1)$	
0	$(2.05969289730 + 1.55594910118i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
0	$(2.05969289730 + 1.55594910118i)10^{-10}$	MBOneLoop + Kira + MBnumerics
10^{-8}	$(2.05969289342 + 1.55594909187i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
10^{-8}	$(2.05969289363 + 1.55594909187i)10^{-10}$	MBOneLoop + Kira + MBnumerics
10^{-4}	$(2.05965609497 + 1.55585605343i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
10^{-4}	$(2.05965609489 + 1.55585605343i)10^{-10}$	MBOneLoop + Kira + MBnumerics

Table 2: *The Feynman integral $J_4(12-2\epsilon,1,5,1,1)$ as defined in (**??**) compared to numbers from [6]. The $I_{4,2222}^{[d+]^4}$ is the scalar integral where propagator 2 has index $\nu_2 = 1+(1+1+1+1) = 5$, the others have index 1. The integral corresponds to D_{1111} in notations of LoopTools [23]. For $x=0$, the Gram determinant vanishes. We see an agreement of about 10 to 11 relevant digits. The deviations of the two calculations seem to stem from a limited accuracy of the Pade approximations used in [6].*

Summary

- **We derived a new recursion relation for 1-loop scalar Feynman integrals:** self-energies, vertices, boxes etc.
  Extremely efficient because delivers low-dimensional Mellin-Barnes integrals.
  The condition $\nu_i = 1$ was essential for that.
  For Mellin-Barnes loop-by-loop integrals: $\nu_i \neq 1$ needed.

  A generalization to multiloops seems to be not straightforward.
- **Solving the recursions in terms of special functions reproduces essential parts of the results of Tarasov et al. from 2003.**
- **Concerning their $b_{3,4}$-terms, we see differences. Their result is not controlled in Minkowskian kinematical situations.**
- **Stable numerics** for Appell $F_!$ and Lauricella-Saran F_S functions.
- **We derived a new series of Mellin-Barnes representations:**
  1-dim. for self-energies, 2-dim. for vertices, and 3-dim. for box diagrams for the most general massive kinematics. Compared to dim=3, 5, 9 respectively, in the "conventional" Mellin-Barnes-approach.
  Again, we see no direct generalization to multi-loops.
  For small Gram determinants these results are stable, even without special adaptations.

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