Scalar 1-loop Feynman integrals in arbitrary space-time dimension D

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[1] Part of talk also held at LL2018: TR, Blumlein, Dr. Phan [1].

Why one-loop Feynman integrals in $D = 4 + 2n - 2\epsilon$ dimensions?

Basics

The seminal papers on 1-loop Feynman integrals:
't Hooft, Veltman, 1978 [2]: "Scalar oneloop integrals"
Passarino, Veltman, 1978 [3]: "One Loop Corrections for e^+e^- Annihilation into $\mu^+\mu^-$ in the Weinberg Model"

Interest in 1-loop integrals from basically two sides

1.
For many-particle calculations, there appear inverse Gram determinants from tensor reductions in the answers.

These $1/G(p_i)$ may diverge, because Gram dets can exactly vanish: $G(p_i) \equiv 0$.

One may perform tensor reductions so that no inverse Grams appear, but one has to buy 1-loop integrals in higher dimensions, $D = 4 + 2n - 2\epsilon$. See [4, 5].

Interest in 1-loop integrals from basically two sides

2.
Higher-order loop calculations need h.o. contributions from ϵ-expansions of 1-loops:
$1/(d-4) = -1/(2\epsilon)$ and $\Gamma(\epsilon) = a/\epsilon + c + \epsilon + \cdots$

A Seminal paper was on ϵ-terms of 1-loop functions:
Nierste, Müller, Böhm, 1992 [6]: "Two loop relevant parts of D-dimensional massive scalar one loop integrals"

Conclusion → 1-loop integrals in D dimensions

A general solution in D dimensions was derived in another 2 seminal papers:
Tarasov, 2000 [7],
Fleischer, Jegerlehner, Tarasov, 2003 [8]: "A New hypergeometric representation of one loop scalar integrals in d dimensions"

I was wondering if the results of Fleischer/Jegerlehner/Tarasov (2003) are useful for deriving numerical black-box software applications?

So we decided to study the issue from scratch in 2 steps:

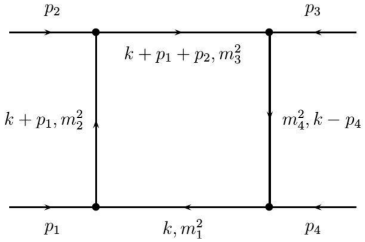
1st step, done: Re-derive analytical expressions for scalar one-loop integrals as meromorphic functions of **arbitrary** space-time dimension D and for **arbitrary** kinematics.

- 2-point functions: Gauss hypergeometric functions ${}_2F_1$ [9]
- 3-point functions: additional Kamp'e de F'eriet functions F_1; there are the Appell functions $F_1, \ldots F_4$ [10]
- 4-point functions: additional Lauricella-Saran functions F_S [11]

2nd step, future: Derive the Laurent expansions around the singular points of these functions at $D = 4, 6, \cdots$.

This talk:

- Analytical expressions for self-energies, vertices, boxes
- Numerical checks



$$J_N \equiv J_N(d;\{p_ip_j\},\{m_i^2\}) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_N^{\nu_N}} \quad (1)$$

with

$$D_i = \frac{1}{(k+q_i)^2 - m_i^2 + i\epsilon}. \quad (2)$$

$$\nu_i = 1, \quad \sum_{i=1}^{n} p_i = 0 \quad (3)$$

$$J_n = (-1)^n \Gamma\left(n - d/2\right) \int_0^1 \prod_{j=1}^n dx_j \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{1}{F_n(x)^{n-d/2}} \quad (4)$$

Here, the F-function is the second Symanzik polynomial.

The F-function is derived from the propagators (2),

$$M^2 = x_1 D_1 + \cdots + x_N D_N = k^2 - 2Qk + J. \quad (5)$$

Using $\delta(1 - \sum x_i)$ under the integral in order to transform linear terms in x into quadratic ones, we may obtain:

$$F_n(x) = -(\sum_i x_i)\, J + Q^2 = \frac{1}{2} \sum_{i,j} x_i Y_{ij} x_j - i\epsilon, \quad (6)$$

The Y_{ij} are elements of the Cayley matrix, introduced for a systematic study of one-loop n-point Feynman integrals e.g. in [12]

$$Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2. \quad (7)$$

There are $N_n = \frac{1}{2} n(n+1)$ different Y_{ij} for n-point functions: $N_3 = 6, N_4 = 10, N_5 = 15$.

The operator $\mathbf{k}^-$ ...

...will reduce an n-point Feynman integral J_n to an $(n-1)$-point integral J_{n-1} by shrinking the propagator $1/D_k$

$$\mathbf{k}^- \; J_n \quad = \quad \mathbf{k}^- \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{\prod_{j=1}^n D_j} \quad = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{\prod_{j \neq k, j=1}^n D_j}. \tag{8}$$

Mellin-Barnes representation

$$\frac{1}{(1+z)^\lambda} \quad = \quad \frac{1}{2\pi i} \int\limits_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(\lambda+s)}{\Gamma(\lambda)} \, z^s \quad = \quad {}_2F_1\left[\begin{array}{c} \lambda, b\,; \\ b\,; \end{array} \; -z \right]. \tag{9}$$

Eqn. (9) is valid if $|\mathrm{Arg}(z)| < \pi$.
The integration contour has to be chosen such that the poles of $\Gamma(-s)$ and $\Gamma(\lambda+s)$ are well-separated. The right hand side of (9) is identified as Gauss' hypergeometric function. For more details see [13]).

F-function and Gram and Cayley determinants

Gram and Cayley det's were introduced by Melrose (1965) [12]. The Cayley determinant $\lambda_{12\ldots N}$ is composed of the
$Y_{ij} = m_i^2 + m_j^2 - (q_i - q_j)^2$ introduced in (7), and its determinant is:

$$\text{Cayley determinant :} \qquad \lambda_n \equiv \lambda_{12\ldots n} \quad = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}. \tag{10}$$

We also define the $(n-1) \times (n-1)$ dimensional Gram determinant $g_n \equiv g_{12\cdots n}$,

$$G_n \equiv G_{12\cdots n} = - \begin{vmatrix} (q_1-q_n)^2 & (q_1-q_n)(q_2-q_n) & \ldots & (q_1-q_n)(q_{n-1}-q_n) \\ (q_1-q_n)(q_2-q_n) & (q_2-q_n)^2 & \ldots & (q_2-q_n)(q_{n-1}-q_n) \\ \vdots & \vdots & \ddots & \vdots \\ (q_1-q_n)(q_{n-1}-q_n) & (q_2-q_n)(q_{n-1}-q_n) & \ldots & (q_{n-1}-q_n)^2 \end{vmatrix}. \quad (1$$

Both determinants are independent of a common shifting of the momenta q_i.
Further, the Gram det G_n is independent of the propagator masses.

Co-factors of the Cayley matrix

One further notation will be introduced, namely that of co-factors of the Cayley matrix.
Also called signed minors in e.g. [12, 14]):

$$\begin{pmatrix} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{pmatrix}_n . \tag{12}$$

The signed minors are determinants, labeled by those rows $j_1, j_2, \cdots j_m$ and columns $k_1, k_2, \cdots k_m$ which have been discarded from the definition of the Cayley determinant $()_n$, with a sign convention.

$$\text{sign} \begin{pmatrix} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{pmatrix}_n = (-1)^{j_1+j_2+\cdots+j_m+k_1+k_2+\cdots+k_m} \times \texttt{Signature}[j_1, j_2, \cdots j_m] \times \texttt{Signature}[k_1, k_2, \cdots k_m] \tag{13}$$

Here, `Signature` (defined like the Mathematica command) gives the sign of permutations needed to place the indices in increasing order.

Cayley matrix, by definition:

$$\lambda_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n . \tag{14}$$

Further, it is (see [15]):

$$-\frac{1}{2}\,\partial_i \lambda_n \equiv -\frac{1}{2}\,\frac{\partial \lambda_n}{\partial m_i^2} = \begin{pmatrix} 0 \\ i \end{pmatrix}_n . \tag{15}$$

Rewriting the F-function further, exploring the $\delta(1-\sum x_i)$ ...

The δ-function: The elimination of x_n, one of the x_i, creates linear terms in $F(x)$.

$$F_n(x) = x^T G_n x + 2H_n^T x + K_n. \quad (16)$$

The $F_n(x)$ may be cast by shifts $x \to (x-y)$ into the form

$$F_n(x) = (x-y)^T G_n (x-y) + r_n - i\varepsilon = \Lambda_n(x) + r_n - i\varepsilon = \Lambda_n(x) + R_n, \quad (17)$$

$$\Lambda_n(x) = (x-y)^T G_n (x-y), \quad (18)$$

and

$$r_n = K_n - H_n^T G_n^{-1} H_n = -\frac{\lambda_n}{g_n} =! -\frac{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n}{()_n}. \quad (19)$$

The inhomogeneity $R_n = r_n - i\varepsilon$ carries the $i\varepsilon$-prescription.

The linear shifts y_i

The $(n-1)$ components y_i of the vector y appearing here in $F_n(x)$ are:

$$y_i = -\left(G_n^{-1} K_n\right)_i, \quad i \neq n \tag{20}$$

The following relations are also valid:

$$y_i = \frac{\partial r_n}{\partial m_i^2} = -\frac{1}{g_n}\frac{\partial \lambda_n}{\partial m_i^2} = -\frac{\partial_i \lambda_n}{g_n} = \frac{2}{g_n}\left(\begin{array}{c} 0 \\ i \end{array}\right)_n, \quad i = 1 \cdots n. \tag{21}$$

The auxiliary condition $\sum_i^n y_i = 1$ is fulfilled.

- The notations for the F-function are finally independent of the choice of the variable which was eliminated by use of the δ-function in the integrand of (4).
- The inhomogeneity R_n is the only variable carrying the causal $i\epsilon$-prescription, while e.g. $\Lambda(x)$ and the y_i are by definition real quantities.

The recursion relation for J_n

One may use the Mellin-Barnes relation (9) in order to decompose the integrand of J_n given in (4) as follows:

$$J_n \sim \int dx \frac{1}{[F(x)]^{n-\frac{d}{2}}} \equiv \int dx \frac{1}{[\Lambda_n(x)+R_n]^{n-\frac{d}{2}}} \equiv \int dx \frac{R_n^{-(n-\frac{d}{2})}}{[1+\frac{\Lambda_n(x)}{R_n}]^{n-\frac{d}{2}}}$$

$$= \int dx \frac{R_n^{-(n-\frac{d}{2})}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s)\,\Gamma(n-\frac{d}{2}+s)}{\Gamma(n-\frac{d}{2})} \left[\frac{\Lambda_n(x)}{R_n}\right]^s, \quad (22)$$

for $|\mathrm{Arg}(\Lambda_n/R_n)| < \pi$. The condition always applies. Further, the integration path in the complex s-plane separates the poles of $\Gamma(-s)$ and $\Gamma(n-\frac{d}{2}+s)$.
As a result of (22), the Feynman parameter integral of J_n becomes homogeneous:

$$\kappa_n = \int dx \left[\frac{\Lambda_n(x)}{R_n}\right]^s$$

$$= \prod_{j=1}^{n-1} \int_0^{1-\sum_{i=j+1}^{n-1} x_i} dx_j \left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \int dS_{n-1} \left[\frac{\Lambda_n(x)}{R_n}\right]^s. \quad (23)$$

The recursion relation for J_n

In order to solve the integral in (23), we consider the differential operator $\hat{P}_n$ [16, 17],

$$\hat{P}_n \left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \sum_{i=1}^{n-1} \frac{1}{2}(x_i - y_i)\frac{\partial}{\partial x_i} \left[\frac{\Lambda_n(x)}{R_n}\right]^s = s \left[\frac{\Lambda_n(x)}{R_n}\right]^s . \quad (24)$$

This eigenvalue relation allows to introduce the operator $\hat{P}_n$ into the integrand of (23):

$$\kappa_n = \int dS_{n-1} \frac{\hat{P}_n}{s} \left[\frac{\Lambda_n(x)}{R_n}\right]^s = \frac{1}{2s} \sum_{i=1}^{n-1} \prod_{k=1}^{n-1} \int_0^{u_k} dx'_k \, (x_i - y_i)\frac{\partial}{\partial x_i} \left[\frac{\Lambda_n(x)}{R_n}\right]^s . \quad (25)$$

After a series of manipulations in order to perform one of the x-integrations – by partial integration, eating the corresponding differential – one arrives at:

$$J_n = \frac{(-1)^n}{2\pi i} \int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(n-\frac{d}{2}+s)\Gamma(s+1)}{2\,\Gamma(s+2)} \left(\frac{1}{R_n}\right)^{n-\frac{d}{2}}$$

$$\times \sum_{i=1}^{n} \left\{ \left(\frac{\partial r_n}{\partial m_i^2}\right) \int dS_{n-2}^{(i)} \left[\frac{F_{n-1}^{(i)}}{R_n} - 1\right]^s \quad (26)$$

We stress again that only the R_n carries an $i\epsilon$.

Now it is important to eliminate the term (-1) from the combination $(F^{(i)}_{n-1}/R_n - 1)^s$ under the Mellin-Barnes integral over s, because then we arrive at a sum over the n different $(n-1)$-point functions arising from skipping a propagator from the original integral. In fact, this may be arranged using the following relation for $(-z) = F/R - 1$ [18]:

$$\int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(a+s)\,\Gamma(b+s)}{\Gamma(c+s)} (-z)^s \tag{27}$$

$$= \int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(a+b-c-s)\Gamma(c-a+s)\Gamma(c-b+s)}{\Gamma(c-a)\Gamma(c-b)} \, (1-z)^{c-a-b+s},$$

provided that $|\mathrm{Arg}(-z)| < 2\pi$.

We arrive at the following recursion relation:

The recursion relation for 1-loop n-point functions

$$J_n(d,\{q_i,m_i^2\}) = \frac{-1}{2\pi i}\int_{-i\infty}^{+i\infty} ds\, \frac{\Gamma(-s)\,\Gamma(\frac{d-n+1}{2}+s)\Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} R_n^{-s} \times \sum_{k=1}^{n}\left(\frac{1}{r_n}\frac{\partial r_n}{\partial m_k^2}\right) \mathbf{k}^- J_n(d+2s;\{q_i,m_i^2\}). \quad (28)$$

The cases $G_n = 0$ and $\lambda_n = r_n = 0$ prevent the use of the Mellin-Barnes transformation.
$\rightarrow$ Perform reductions to simpler functions a la [8].

1-point function, or tadpole

$$J_1(d;m^2) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2-m^2+i\epsilon} = -\frac{\Gamma(1-d/2)}{(m^2-i\epsilon)^{1-d/2}}. \quad (29)$$

Comments

- In Tarasov 2003 [8], a recursion was derived where our Mellin-Barnes integral is replaced by an infinite sum to be solved. Formulae for 2,3,4-point functions are given.
- Any 4-point integral e.g. is a **3-fold** integral.
  With AMBRE, we get up to **9-fold** integrals for e.g. box integrals, instead!
- Euklidean and Minkoswkian integrals converge equally good. See J. Usovitsch's talk at LL2018 [19].
- No Gram=0 problem. See last section and Usovitsch, TR [20].

The 2-point function

From our recursion relation (28), taken at $n = 2$ and using the expression (29) with $d \to d + 2s$ for the one-point functions under the integral, one gets the following representation:

$$J_2(D; q_1, m_1^2, q_2, m_2^2) = \frac{e^{\epsilon\gamma_E}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma\left(\frac{D-1}{2}+s\right)\,\Gamma(s+1)}{2\,\Gamma\left(\frac{D-1}{2}\right)} R_2^s \times \left[\frac{1}{r_2}\frac{\partial r_2}{\partial m_2^2}\frac{\Gamma\left(1-\frac{D+2s}{2}\right)}{(m_1^2)^{1-\frac{D+2s}{2}}} + (m_1^2 \leftrightarrow m_2^2)\right]. \quad (30)$$

One may close the integration contour of the MB-integral in (30) to the right, apply the Cauchy theorem and collect the residua originating from two series of zeros of arguments of Γ-functions at $s = m$ and $s = m - d/2 - 1$ for $m \in \mathbb{N}$.

The first series stems from the MB-integration kernel, the other one from the dimensionally shifted 1-point functions.

And then summing up in terms of Gauss' hypergeometric functions.

The 2-point function (slightly rewritten), $R_2 \equiv R_{12}$

$$J_2(d;Q^2,m_1^2,q_2,m_2^2) = -\frac{\Gamma\left(2-\frac{d}{2}\right)}{(d-2)}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}\right)}\frac{\partial_2 R_2}{R_2} \tag{31}$$

$$\left[(m_1^2)^{\frac{d}{2}-1}\,{}_2F_1\left[\begin{matrix}1, \frac{d}{2}-\frac{1}{2}; \\ \frac{d}{2};\end{matrix} \; \frac{m_1^2}{R_2}\right] + \frac{R_2^{\frac{d}{2}-1}}{\sqrt{1-\frac{m_1^2}{R_2}}}\sqrt{\pi}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)}\right]$$

$$+ \;(m_1^2 \leftrightarrow m_2^2)$$

The representation (31) is valid for $\left|\frac{m_1^2}{r_{12}}\right| < 1$, $\left|\frac{m_2^2}{r_{12}}\right| < 1$ and $\mathcal{R}\text{e}(\frac{d-2}{2}) > 0$.

The result is in agreement with Eqn. (53) of of Tarasov et al. (2003) [8].

The 3-point function

According to the master formula (28), we can write the massive 3-point function as a sum of three terms:

$$J_3 \quad = \quad J_{123} + J_{231} + J_{312}, \tag{32}$$

using the representation for e.g. J_{123}

$$\begin{aligned} J_{123}(d, \{q_i, m_i^2\}) \quad &= \quad -\frac{e^{\epsilon\gamma_E}}{2\pi i} \int\limits_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(\frac{d-2+2s}{2})\Gamma(s+1)}{2\,\Gamma(\frac{d-2}{2})} R_3^{-s} \\ &\times \frac{1}{r_3} \frac{\partial r_3}{\partial m_3^2} \; J_2(d+2s; q_1, m_1^2, q_2, m_2^2). \end{aligned} \tag{33}$$

Here, $J_2(d+2s; q_1, m_1^2, q_2, m_2^2)$ is given by (31), taken at $d+2s$ dimensions. By performing the Mellin-Barnes integrals, one gets three terms, each consisting of eight series, from taking the residues by closing the integration contours to the right; one of the three terms is:

$J_3 = J_{123} + J_{231} + J_{312}$ is, with $R_3 = R_{123}, R_2 = R_{12}$ etc. here:

The massive vertex:

$$
\begin{aligned}
J_{123} \quad = \quad & \Gamma\left(2-\frac{d}{2}\right) \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{r_2}{2\sqrt{1-m_1^2/r_2}} \\
& \left[- R_2^{d/2-2} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)} \, {}_2F_1\left[\begin{matrix} \frac{d-2}{2}, 1; & \frac{R_2}{R_3} \\ \frac{d-1}{2}; & \end{matrix} \right] + R_3^{d/2-2} \, {}_2F_1\left[\begin{matrix} 1, 1; & \frac{R_2}{R_3} \\ 3/2; & \end{matrix} \right] \right] \\
& + \Gamma\left(2-\frac{d}{2}\right) \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{m_1^2}{4\sqrt{1-m_1^2/r_2}} \\
& \left[+ \frac{2(m_1^2)^{d/2-2}}{d-2} F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2}\right) - R_3^{d/2-2} F_1\left(1; 1, \frac{1}{2}; 2; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2}\right) \right] \\
& + \quad (m_1^2 \leftrightarrow m_2^2)
\end{aligned}
$$

For $d \to 4$, both the $[...]$ approach zero.

So the J_3 is finite in this limit, as it should be for a massive 3-point function.

The $\partial_i \lambda_{j\ldots}$ is defined in (21). The representation (32) is valid for Re$(d-2/2) > 0$. The conditions $|m_i^2/R_{ij}| < 1$, $|R_{ij}/R_{ijk}| < 1$ had to be met during the derivation.
The result may be analytically continued in a straightforward way, however, in the complete complex domain.

The functions ${}_2F_1$ and F_1 of the b_{ijk}-boundary terms are met
by setting $d = 4$ in the corresponding functions J_{ijk}.

For the 3-point function, we look at the expression $J_{123} + J_{231} + J_{312}$.

We should agree with Eqn. (74) to (76) of Tarasov (2003).

Our terms with d-dimensional F_1 and ${}_2F_1$ do agree exactly, but $(b_{123} + b_{231} + b_{312})$ looks quite different.

Tarasov (2003) [8], Eqns. (73) and (75)

There are kinematic conditions on internal momenta $q_{ij}^2 = (q_i - q_j)^2$ to be respected; the b_3-term of Tarasov becomes:

$$J_3(b_3) = \theta(-G_3) \times \theta(q_{ij}^2) \times \theta(\frac{m_i^2}{r_3} - 1) \times \frac{\Gamma(2-d/2)}{\lambda_3} \left(2^{3/2}\,\pi\,\sqrt{-G_3}\;R_3{}^{d/2-1}\right) \quad (34)$$

Otherwise:

$$J_3(b_3) = b_3 = 0. \quad (35)$$

Numerics for 3-point functions, table 1

$[p_i^2]$, $[m_i^2]$	[+100, +200, +300], [10, 20, 30]	
G_{123} λ_{123} m_i^2/r_{123} m_i^2/r_{12} m_i^2/r_{23} m_i^2/r_{31} $\sum J$-terms	–160000 –8860000 –0.180587, –0.361174, –0.541761 –0.97561, –1.95122, –2.92683 –0.39801, –0.79602, –1.19403 –0.180723, –0.361446, –0.542169 (0.019223879 – 0.007987267 I)	
$\sum b_3$-terms J_3(TR)	0 (0.019223879 – 0.007987267 I)	
b_3-term $b_3 + \sum J$-terms	(–0.089171509 + 0.069788641 I) (–0.012307377 – 0.009301346 I)	+ (0.022214414)/eps
J_3(OT)	$\sum J$-terms, b_3-term $\to$ 0, OK	
MB suite		
(-1)*fiesta3	-(0.012307 + 0.009301 I)	+ (8*10-6 + 0.00001 I) pm4)
LoopTools/FF, ϵ^0	0.0192238790286244077-0.00798726725497102795 i	

Table 1: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. Red input quantities (external momenta shown here!) suggest that, according to Eqn. (73) in Tarasov (2003) [8], one has to set $b_3 = 0$.

Although b_3 of [8] deviates from our vanishing value, it has to be set to zero, $b_3 \to 0$.

The results of both calculations for J_3 agree for this case.

Numerics for 3-point functions, table 2

$[p_i^2]$, $[m_i^2]$	[-100, +200, -300], [10, 20, 30]	
G_{123} λ_3 m_i^2/r_3 m_i^2/r_{12} m_i^2/r_{23} m_i^2/r_{31} $\sum J$-terms	480000 -19300000 0.248705, 0.497409, 0.746114 0.248447, 0.496894, 0.745342 -0.39801, -0.79602, -1.19403 0.104895, 0.20979, 0.314685 (-0.012307377 - 0.056679689 I)	 + (+ 0.012825498 I)/eps
$\sum b_3$-terms J_3(TR)	(+ 0.047378343 I) (-0.012307377 - 0.009301346 I)	- (+ 0.012825498 I)/eps
b_3-term $b_3 + \sum J$-terms	(+ 0.047378343 I) (-0.012307377 - 0.009301346 I)	- (+ 0.012825498 I)/eps
J_3(OT)	$\sum J$-terms, b_3-term$\to$0, **gets wrong!**	
MB suite		
(-1)*fiesta3	-(0.012307 + 0.009301 I)	+ (8*10-6 + 0.00001 I) pm4)
LoopTools/FF, ϵ^0	-0.012307377367782063O - 0.0093013461700863289 i	

Table 2: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. Red input quantities suggest that, according to eq. (73) in Tarasov2003 [8], one has to set $b_3 = 0$. Further, we have set in the numerics for eq. (75) of Tarasov2003 [8] that Sqrt[-g123 + I*epsil], what looks counter-intuitive for a "momentum"-like function.

Both results agree if we *do not* set Tarasov's $b_3 \to 0$.

Numerics for 3-point functions, table 3

p_i^2	–100,–200,–300	
m_i^2	10,20,30	
G_{123}	–160000	
λ_{123}	15260000	
m_i^2/r_{123}	0.104849, 0.209699, 0.314548	
m_i^2/r_{12}	0.248447, 0.496894, 0.745342	
m_i^2/r_{23}	0.133111, 0.266223, 0.399334	
m_i^2/r_{31}	0.104895, 0.20979, 0.314685	
$\sum J$-terms	(0.0933877 – 0 I)	– (0.0222144 – 0 I)/eps
$\sum b$-terms	-0.101249	+ 0.0222144/eps
J_3(TR)	(–0.00786155 – 0 I)	
b_3	(-0.101249 + 0 I)	+ (0.0222144 + 0 I)/eps
b_3+J-terms	(–0.007861546 + 0 I)	
J_3(OT)	b_3+J-terms $\rightarrow$ OK	
MB suite	–0.007862014, 5.002549159*10-6, 0	
(-1)*fiesta3	–(0.007862)	+ (6*10-6 + 6*10-6 I pm10)
LoopTools/FF, ϵ^0	–0.00786154613229082290	

Table 3: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$.
Agreement with Tarasov (2003).

The 4-point function

According to the master formula (28), we can write the massive 4-point function as a sum of four terms:

$$J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}, \tag{36}$$

Each of the four terms has the structure

$$J_{1234} = \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d-3}{2}\right)} \times (r_{1234})^{\frac{d}{2}-2} \times \hat{b}_{1234} + \Gamma\left(2-d/2\right) \times \hat{J}^{d}_{1234} \tag{37}$$

The pre-factor is singular: $\Gamma(2-d/2) = 1/\epsilon + \cdots$ for $d \geq 4-2\epsilon$.

We agree for $\hat{J}^{d}_{1234}$ etc. with Tarasov (2003) [8].
For the b_4-term, the situation is a bit unclear.

The boundary term $\hat{b}_{1234}$ is independent of d:

$$
\begin{aligned}
\hat{b}_{1234} \quad = \quad & \frac{1}{2}\left(\frac{b_{123}}{r_{1234}}\frac{\partial r_{1234}}{\partial m_4^2}\right)\frac{\sqrt{\pi}}{\sqrt{1-r_{123}/r_{1234}}} \\
& + \sqrt{\pi}\left(\frac{1}{r_{1234}}\frac{\partial r_{1234}}{\partial m_4^2}\right)\left(\frac{1}{r_{123}}\frac{\partial r_{123}}{\partial m_3^2}\right)\left(\frac{1}{4g_{12}}\right)\times \\
& \times\left[\frac{\partial_2\lambda_{12}}{\sqrt{1-m_1^2/r_{12}}}+\frac{\partial_1\lambda_{12}}{\sqrt{1-m_2^2/r_{12}}}\right]\left(\frac{1}{\sqrt{1-r_{12}/r_{123}}}\right)F_1\left(\frac{1}{2};1,\frac{1}{2};\frac{3}{2};\frac{r_{12}}{r_{1234}},\frac{r_{12}}{r_{123}}\right) \quad (38) \\
& + \sqrt{\pi}\left(\frac{1}{r_{1234}}\frac{\partial r_{1234}}{\partial m_4^2}\right)\left(\frac{1}{r_{123}}\frac{\partial r_{123}}{\partial m_3^2}\right)\times \\
& \times\left[\left(\frac{\partial_2\lambda_{12}}{1-\frac{m_1^2}{r_{12}}}\right)\left(\frac{m_1^2}{8\lambda_{12}}\right)\left(\frac{r_{123}}{r_{123}-m_1^2}\right)\right. \\
& \left.\times F_S\left(\frac{1}{2},1,1;1,1,\frac{1}{2};2,2,2;\frac{m_1^2}{r_{1234}},\frac{m_1^2}{m_1^2-r_{123}},\frac{m_1^2}{m_1^2-r_{12}}\right)+(1\leftrightarrow 2)\right] \\
& +(2,3,1)+(3,1,2).
\end{aligned}
$$

The boundary term b_4 has not been exactly defined in [8], concerning the kinematical conditions. We did not perform massive numerical tests.

An alternative writing of $J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}$ is, with $R_4 = R_{1234}, R_3 = R_{123}, R_2 = R_{12}$ etc. here:

The massive box function

$$
\begin{aligned}
J_{1234} \quad = \quad & \Gamma\left(2-\frac{d}{2}\right)\frac{\partial_4 r_4}{r_4}\Bigg\{ \\
& \left[\frac{b_{123}}{2}\left(-R_3^{d/2-2}\,{}_2F_1\left[\begin{matrix}\frac{d-3}{2},1; & \frac{R_2}{R_3}\\ \frac{d}{2}-1; & \end{matrix}\right]+R_4^{d/2-2}\sqrt{\pi}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}\,{}_2F_1(d\to 4)\right)\right] \\
& +\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}\frac{\sqrt{\pi}}{4}\frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{\sqrt{1-m_1^2/R_2}}\,{}_2F_1\left[\begin{matrix}1/2,1; & \frac{R_2}{R_3}\\ 1; & \end{matrix}\right] \\
& \left[+\frac{R_2^{d/2-2}}{d-3}F_1\left(\frac{d-3}{2};1,\frac{1}{2};\frac{d-1}{2};\frac{R_2}{R_4},\frac{R_2}{R_3}\right)-R_4^{d/2-2}F_1\left(d\to 4\right)\right] \\
& \frac{m_1^2}{8}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}\frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{r_2}\frac{r_3}{r_3-m_1^2}\frac{r_2}{r_2-m_1^2} \\
\Bigg[-(m_1^2)^{d/2-2}\frac{\Gamma\left(\frac{d}{2}-3/2\right)}{\Gamma\left(\frac{d}{2}\right)} & F_S\left(d/2-3/2,1,1,1,1,d/2,d/2,d/2,d/2,\frac{m_1^2}{R_4},\frac{m_1^2}{m_1^2-R_3},\frac{m_1^2}{m_1^2-R_2}\right) \\
& +R_4^{d/2-2}\sqrt{\pi}F_S(d\to 4)\Bigg] \quad + \quad (m_1^2\leftrightarrow m_2^2)\Bigg\}
\end{aligned}
\tag{39}
$$

For $d \to 4$,all three $[\ldots]$ approach zero.
So that the massive J_4 gets finite then: OK.

The cases of vanishing Cayley determinant and vanishing Gram determinant

We refer to two important special cases, where the general derivations cannot be applied.
In the case of vanishing Cayley determinant, $\lambda_n = 0$, we cannot introduce the inhomogenei $R_n = -\lambda_n / G_n$ into the Symanzik polynomial F_2. Let us assume that it is $G_n \neq 0$, so that $r_n = 0$. A useful alternative representation to (28) is known from the literature see e.g. Eqn. (3) in [8]:

$$J_n(d) = \frac{1}{d-n-1} \sum_{k=1}^{n} \frac{\partial_k \lambda_n}{G_n} \mathbf{k}^- J_n(d-2). \quad (40)$$

Another special case is a vanishing Gram determinant, $G_n = 0$. Here, again one may use Eqn. (3) of [8] and the result is (for $\lambda_n \neq 0$):

$$J_n(d) = -\sum_{k=1}^{n} \frac{\partial_k \lambda_n}{2\lambda_n} \mathbf{k}^- J_n(d). \quad (41)$$

The representation was, for the special case of the vertex function, also given in Eqn. (46) of [21].

Example: A massive 4-point function with vanishing Gram determinant I

As a very interesting, non-trivial example we study the numerics of a massive 4-point function with a small or vanishing Gram determinant [20]. The example has been taken from Appendix C of [5]. The kinematics is:

$$
\begin{aligned}
p_1^2 = p_2^2 &= m_1^2 = m_3^2 = m_4^2 = 0, \\
m_2^2 &= (911876/10000)^2, \\
p_3^2 &= s_3 = s_{\nu u} = 10000, \\
p_4^2 &= t_{ed} = -60000(1+x), \\
s &= (p_1+p_2)^2 = s_{12} = t_{e\mu} = -40000, \\
t &= (p_2+p_3)^2 = s_{23} = s_{\mu\nu u} = 20000.
\end{aligned}
\tag{42}
$$

The resulting Gram determinant is

$$
G_4 = -2t_{e\mu}[s_{\mu\nu u}^2 + s_{\nu u}t_{ed} - s_{\mu\nu u}(s_{\nu u} + t_{ed} - t_{e\mu})]. \tag{43}
$$

Example: A massive 4-point function with vanishing Gram determinant II

The Gram determinant vanishes if

$$t_{ed} \to t_{ed,crit} = \frac{s_{\mu\nu u}(s_{\mu\nu u} - s_{\nu u} + t_{e\mu})}{s_{\mu\nu u} - s_{\nu u}}. \quad (44)$$

Introducing a parameter x, we can describe the vanishing of G_4 by the limiting process $x \to 0$:

$$t_{ed} = (1+x)t_{ed,crit}, \quad (45)$$
$$G_4 = -2xs_{\mu\nu u}t_{e\mu}(s_{\mu\nu u} - s\nu u + t_{e\mu}). \quad (46)$$

$x = 0$ $\to$ the Gram determinant vanishes. Further, it is simple to calculate e.g. the value at $x = 1$:
$G_4(x=1) = -4.8 \times 10^{13}$ GeV^3.
Further, if the Gram determinant vanishes exactly, a reduction of J_4 according to (1) is possible and allows a simple and very precise calculation.
For small x, the calculations become unstable with usual reductions a la Passarino/Veltman [3], due to the occurrence of inverse Gram determinants.

Example: A massive 4-point function with vanishing Gram determinant III

Several solutions were worked out; we refer to the review [22].
Here we will use for comparisons the solution which was worked out in [5] and has been implemented in the C++ program PJFry [23, 24, 25, 26].

In the new approach presented here, a calculation of J_4 is possible as follows.
First reduce the indices ν_i of the propagators, if any, to the canonical value $\nu_i = 1$, and then apply the MB-formula directly.
This has been done with the C++ package KIRA [27], without generating inverse powers of Gram determinants.
In fact, the procedure introduces for a $J_4(d)$ a basis of functions $J_4(d+2n)$, where $n \geq 0$ is related to the indices $\nu_i \geq 1$. We use here the short notation

$$J_4(D,\nu_1,\nu_2,\nu_3,\nu_4) = I_4(D,p_1^2,p_2^2,p_3^2,p_4^2,s,t,m_1^2,m_2^2,m_3^2,m_4^2)[\nu_1,\nu_2,\nu_3,\nu_4]. \quad (47)$$

The numerical solution of of the Mellin-Barnes integral for J_4 is numerically stable also in the Minkowskian case. This contrasts the usual MB-representations derived with AMBRE. A reason is that the instabilities in the AMBRE-approach origin from Γ-functions from Beta-functions which do not appear here.
And finally we reproduce the box integral, dependent on d and the internal variables $\{d, q_1, m_1^2, \cdots q_4, m_4^2\}$ or, equivalently, on a set of external variables, e.g. $\{d, \{p_i^2\}, \{m_i^2\}, s, t\}$:

$$
\begin{aligned}
J_4(d;\{p_i^2\},s,t,\{m_i^2\}) = & \left(\frac{-1}{4\pi i}\right)^4 \frac{1}{\Gamma(\frac{d-3}{2})} \sum_{k_1,k_2,k_3,k_4=1}^{4} D_{k_1k_2k_3k_4} \left(\frac{1}{r_4}\frac{\partial r_4}{\partial m_{k_4}^2}\right) \\
& \left(\frac{1}{r_{k_3k_2k_1}}\frac{\partial r_{k_3k_2k_1}}{\partial m_{k_3}^2}\right)\left(\frac{1}{r_{k_2k_1}}\frac{\partial r_{k_2k_1}}{\partial m_{k_2}^2}\right)(m_{k_1}^2)^{d/2-1} \qquad (48) \\
& \int_{-i\infty}^{+i\infty} dz_4 \int_{-i\infty}^{+i\infty} dz_3 \int_{-i\infty}^{+i\infty} dz_2 \left(\frac{m_{k_1}^2}{R_4}\right)^{z_4}\left(\frac{m_{k_1}^2}{R_{k_3k_2k_1}}\right)^{z_3}\left(\frac{m_{k_1}^2}{R_{k_2k_1}}\right)^{z_2} \\
& \Gamma(-z_4)\Gamma(z_4+1)\frac{\Gamma(z_4+\frac{d-3}{2})}{\Gamma(z_4+\frac{d-2}{2})}\Gamma(-z_3)\Gamma(z_3+1)\frac{\Gamma(z_3+z_4+\frac{d-2}{2})}{\Gamma(z_3+z_4+\frac{d-1}{2})} \\
& \Gamma(z_2+z_3+z_4+\frac{d-1}{2})\Gamma(-z_2-z_3-z_4-\frac{d+2}{2})\Gamma(-z_2)\Gamma(z_2+1).
\end{aligned}
$$

The representation (48) can be treated by the Mathematica packages MB and MBnumerics of the MBsuite, replacing AMBRE by a derivative of MBnumerics [19].

The table contains our numerical result, which was also studied in [5]. In LoopTools notations, numbers correspond to the tensor coefficient D_{2222}.
We have a numerical agreement of more than 10 digits, although we performed here no expansion in the small parameter x. Such an expansion would improve calculations considerably.
Our results give an impression on the accuracy of the small Gram expansion in [5], where an error propagation of the Pade approach was not done: In all cases, [5] had at least 10 reliable digits.

$J_4(12-2\epsilon,1,5,1,1) \to I_{4,2222}^{[d+]^4} = D_{1111}$

x	value for $4! \times J_4(12-2\epsilon,1,5,1,1)$	
0	$(2.05969289730 + 1.55594910118i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
0	$(2.05969289730 + 1.55594910118i)10^{-10}$	MBOneLoop + Kira + MBnumerics
10^{-8}	$(2.05969289342 + 1.55594909187i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
10^{-8}	$(2.05969289363 + 1.55594909187i)10^{-10}$	MBOneLoop + Kira + MBnumerics
10^{-4}	$(2.05965609497 + 1.55585605343i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
10^{-4}	$(2.05965609489 + 1.55585605343i)10^{-10}$	MBOneLoop + Kira + MBnumerics

Table 4: *The Feynman integral $J_4(12-2\epsilon,1,5,1,1)$ as defined in (47) compared to numbers from [5]. The $I_{4,2222}^{[d+]^4}$ is the scalar integral where propagator 2 has index $\nu_2 = 1+(1+1+1+1) = 5$, the others have index 1. The integral corresponds to D_{1111} in notations of LoopTools [28]. For $x = 0$, the Gram determinant vanishes. We see an agreement of about 10 to 11 relevant digits. The deviations of the two calculations seem to stem from a limited accuracy of the Pade approximations used in [5].*

Summary

- **We derived a new recursion relation for 1-loop scalar Feynman integrals:** self-energies, vertices, boxes etc.
  The condition $\nu_i = 1$ seems to be essential for that.
  A generalization to multiloops seems to be not straightforward or impossible.
- **Solving the recursions in terms of special functions reproduces essential parts of the results of Tarasov et al. from 2003.**
- **Concerning their b_3-terms, we see an improvement compared to their paper. Maybe their result is not controlled in some kinematical situations.** Our conclusions concerning that depend somewhat on an interpretation of their text.
- **We derived a new series of Mellin-Barnes representations:**
  1-dim. for self-energies, 2-dim. for vertices, and 3-dim. for box diagrams for the most general massive kinematics. Compared to dim=3, 5, 9 respectively, in the “conventional” Mellin-Barnes-approach.
  Again, we see no direct generalization to multi-loops.
- The special case of **vanishing Gram determinant** $G_n = 0$ is not covered.
  But for small Gram determinants results are stable.

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