Contractions of 1-loop 5-point tensor Feynman integrals

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http://indico.ihep.ac.cn/conferenceOtherViews.py?view=standard&confId=2813

Introduction

Recent more detailed overviews of our approach to tensor reduction:

J. Fleischer, at:
"Frontiers in Perturbative Quantum Field Theory"
10-12 September 2012, Bielefeld University, Germany
http://www2.physik.uni-bielefeld.de/fileadmin/user_upload/workshops/fleischer.pdf

T. Riemann, at:
4th Workshop "HP2 – High Precision for Hard Processes"
4-7 September 2012, Max Planck Institute for Physics, Munich, Germany
https://indico.mpp.mpg.de/contributionDisplay.py?contribId=27&confId=1369

T. Riemann, at:
5th Helmholtz International Summer School - Workshop
Dubna International Advanced School of Theoretical Physics - DIAS TH
Calculations for Modern and Future Colliders
July 23 - August 2, 2012, Dubna, Russia
http://theor.jinr.ru/ calc2012/

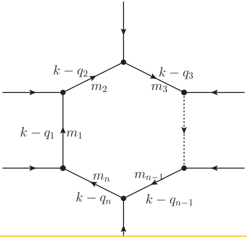
Definitions

n-point tensor integrals of rank R: (n,R)-integrals

$$I_n^{\mu_1\cdots\mu_R} = \int \frac{d^dk}{i\pi^{d/2}} \frac{\prod_{r=1}^{R} k^{\mu_r}}{\prod_{j=1}^{n} c_j^{\nu_j}},$$

$d = 4 - 2\epsilon$ and denominators c_j have *indices* ν_j and *chords* q_j

$$c_j = (k - q_j)^2 - m_j^2 + i\varepsilon$$



tensor integrals due to, e.g.:

- fermion propagators
- three-gauge boson couplings

A simple example

1-loop self-energy:

$$I_2^\mu = \int \frac{d^dk}{i\pi^{d/2}} \frac{k^\mu}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]}$$

$$\textbf{Ansatz}: \quad I_2^\mu = p_\mu \cdot B_1(p, M_1, M_2)$$

Solve:

$$p_\mu \cdot I_2^\mu = p^2 \cdot B_1(p, M_1, M_2)$$

$$= \int \frac{d^dk}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = \int \frac{d^dk}{i\pi^{d/2}} \frac{pk}{D_1\ D_2}$$

$$= \int \frac{d^dk}{i\pi^{d/2}} \left[\frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1\ D_2} \right],$$

$$B_1(p, M_1, M_2) = \frac{1}{2p^2} \left[A_0(M_1) - A_0(M_2) - (p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right]$$

A tensor Feynman integral may be expressed in terms of scalar Feynman integrals.

Passarino-Veltman algorithm

1. Contract n-point and R-rank Feynman integral with *external momenta* p_i^{μ} and with $g^{\mu\nu}$, and cancel propagators
2. Invert the resulting system of linear equations
3. The result consists of $(n-1)$-point and $(R-1)$-rank functions

Reducing tensor rank introduces inverse Gram determinant:

$$I_5^{\mu_1\ldots\mu_{R-1}\mu_R} = \sum_{i=1}^{5} \frac{q_i^{\mu_R}}{\det(G_5)} \left[A_{0i}\ I_5^{\mu_1\ldots\mu_{R-1}} - \sum_{s=1}^{5} A_{si}\ I_4^{\mu_1\ldots\mu_{R-1},s} \right]$$

Gram determinant G_n:

$$G_n = |2q_iq_j|,\ i,j = 1,\ldots n-1 \tag{1}$$

and A_{0i}, A_{si} are kinematic coefficients. The q_i are **internal** momenta.

Systematic approach to tensor reductions:

1,2,3,4-point functions:

- Passarino, Veltman 1978 [1]

~~Open source~~ Source-open programs for 5,6-point reductions:

- LoopTools/FF ($n \leq 5$, $rank \leq 4$), T. Hahn [2, 3] 1998,1990.
- Golem95 T. Binoth et al. [4] 2008
- PJFry ($n \leq 5$, $rank \leq 5$), V. Yundin et al. [5, 6] July 2011

Need in addition a library of scalar functions:

- 't Hooft, Veltman 1979 [7]
- LoopTools/FF T. Hahn [2, 3] 1998,1990
- QCDloop/FF K. Ellis and G. Zanderighi [8, 3] 2007,1990
- OneLOop (complex masses) van Hameren [9] 2010

This talk: Efficient reduction formulae in the algebraic Davydychev-Tarasov-Fleischer-Jegerlehner-TR approach

- Get $n > 4$ tensor reduction with $\cdots$:
- $\cdots$ arbitrary masses
- $\cdots$ inverse pentagon Gram determinants killed
- $\cdots$ full kinematics treated, also with small inverse sub-diagram Gram determinants

- **new:** $\cdots$ multiple sums over tensor coefficients made efficient by **contracting with external momenta**

  Fleischer, TR [10] PLB 701(2011)646 + further simplifications

  Programs:

  OLEC (C++), J. Fleischer, J. Gluza, M. Gluza, TR [11]

  CONTRACTIONS (F95), Andrea Almasy, J. Fleischer, TR [12]

- $\cdots$ higher n point functions, $n \geq 7$ Fleischer, TR [13] PLB 707(2012)375

  Programs: to be done

History of the Approach - not a complete list of references

[14] Melrose 1965: Reduction of Feynman diagrams and Cayley determinants

[15] Davydychev 1991: Integrals in different space-time dimension.

[16] See also Bern et al. 1993

[17] Tarasov 1996: Dimensional recurrence relations

[18] Fleischer,Jegerlehner,Tarasov 2000: 1-loop reductons and signed minors.

[4] Binoth,Guillet,Heinrich,Pilon,Schubert, 2005: Algebraic/numerical formalism for one-loop multi-leg amplitudes

[19] Fleischer and T.Riemann (since 2007) 2011: Complete reduction of 1-loop tensors.

See also Diakonidis et al. [20] 2009 and [21] 2009

[22] Yundin's package PJFry 2011; https://github.com/Vayu/PJFry.

See also Fleischer, TR, Yundin [5, 6]

[10] Fleischer and T.Riemann 2011: Contracted tensor Feynman integrals.

[23] Fleischer and T.Riemann 2012: A solution for tensor reduction of one-loop n-point functions with $n \geq 6$

Tensor integrals expressed in terms of scalar integrals in higher dimensions $D = d + 2l \; = \; 4 - 2\epsilon, \; 6 - 2\epsilon, \cdots$ [Davydychev:1991] , also [Fleischer et al.:2000]

$n_{ij} = \nu_{ij} = 1 + \delta_{ij}, n_{ijk} = \nu_{ij}\nu_{ijk}, \nu_{ijk} = 1 + \delta_{ik} + \delta_{jk}$

$$I_n^{\mu} = \int^d k^{\mu} \prod_{r=1}^{n} c_r^{-1} = -\sum_{i=1}^{n} q_i^{\mu} \, I_{n,i}^{[d+]}$$

$$I_n^{\mu\,\nu} = \int^d k^{\mu}\, k^{\nu} \prod_{r=1}^{n} c_r^{-1} = \sum_{i,j=1}^{n} q_i^{\mu}\, q_j^{\nu}\, n_{ij} \, I_{n,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu} \, I_n^{[d+]}$$

$$I_n^{\mu\,\nu\,\lambda} = \int^d k^{\mu}\, k^{\nu}\, k^{\lambda} \prod_{r=1}^{n} c_r^{-1} = -\sum_{i,j,k=1}^{n} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda}\, n_{ijk} \, I_{n,ijk}^{[d+]^3} + \frac{1}{2} \sum_{i=1}^{n} g^{[\mu\nu} q_i^{\lambda]} \, I_{n,i}^{[d+]^2}$$

$$I_n^{\mu\,\nu\,\lambda\,\rho} = \int \frac{d^d k}{i\pi^{d/2}} \frac{k^{\mu}\, k^{\nu}\, k^{\lambda}\, k^{\rho}}{\prod_{r=1}^{n} c_r} = \sum_{i,j,k,l=1}^{n} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda}\, q_l^{\rho}\, n_{ijkl} \, I_{n,ijkl}^{[d+]^4}$$
$$- \frac{1}{2} \sum_{i,j=1}^{n} g^{[\mu\nu} q_i^{\lambda} q_j^{\rho]}\, n_{ij} I_{n,ij}^{[d+]^3} + \frac{1}{4} g^{[\mu\nu} g^{\lambda\rho]} \, I_n^{[d+]^2} \quad (2)$$

$$
\begin{aligned}
I_n^{\mu\,\nu\,\lambda\,\rho\,\sigma} &= \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu\, k^\nu\, k^\lambda\, k^\rho\, k^\sigma}{\prod_{j=1}^n c_j} \\
&= -\sum_{i,j,k,l,m=1}^{n} q_i^\mu\, q_j^\nu\, q_k^\lambda\, q_l^\rho\, q_m^\sigma\, n_{ijklm}\, I_{n,ijklm}^{[d+]^5} \\
&\quad + \frac{1}{2}\sum_{i,j,k=1}^{n} g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]}\, n_{ijk}\, I_{n,ijk}^{[d+]^4} - \frac{1}{4}\sum_{i=1}^{n} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]}\, I_{n,i}^{[d+]^3}.
\end{aligned}
$$

The integrals are defined in $[d+]^l = 4 - 2\epsilon + 2l$ dimensions.

$I_{n-1,ab}^{\{\mu_1,\cdots\},s}$, $a, b \neq s$ is obtained from $I_n^{\{\mu_1,\cdots\}}$

by

- shrinking line s
- raising the powers of inverse propagators a, b.

Notations: Gram and modified Cayley determinant, signed minors

[Melrose:1965]

Gram determinant G_n:

$$G_n = |2q_iq_j|, i,j = 1,\dots n-1 \tag{3}$$

Modified Cayley determinant $()_N$ of a diagram with N internal lines and chords q_j:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix} \tag{4}$$

with the matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i,j = 1\dots N) \tag{5}$$

The propagators are: $c_i = (k - q_i)^2 - m_i^2$

For the choice $q_n = 0$, both determinants are related:

$$()_N = -G_N$$

$\Rightarrow$ The modfied Cayley determinant $()_N$ does not depend on masses.

Notations: signed minors [Melrose:1965]

signed minors of $()_N$ are constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \equiv$$
$$\equiv (-1)^{\sum_l (j_l+k_l)} \mathrm{sgn}_{\{j\}}\, \mathrm{sgn}_{\{k\}} \begin{vmatrix} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{vmatrix} \qquad (6)$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Example:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_N \equiv \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1N} \\ Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{vmatrix}, \qquad (7)$$

Tarasov's dimensional recurrences for scalars

Following [Tarasov:1996 [17], Fleischer:1999 [18]]
apply recurrence relations, relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $[d+]^l = 4 - 2\epsilon + 2l$:

shift dimension + index:

$$\nu_j (\mathbf{j}^+ I_5^{[d+]}) = \frac{1}{()_5} \left[-\binom{j}{0}_5 + \sum_{k=1}^{5} \binom{j}{k}_5 \mathbf{k}^- \right] I_5 \tag{8}$$

shift dimension:

$$(d - \sum_{i=1}^{5} \nu_i + 1) I_5^{[d+]} = \frac{1}{()_5} \left[\binom{0}{0}_5 - \sum_{k=1}^{5} \binom{0}{k}_5 \mathbf{k}^- \right] I_5, \tag{9}$$

shift index:

$$\nu_j \mathbf{j}^+ I_5 = \frac{1}{\binom{0}{0}_5} \sum_{k=1}^{5} \binom{0j}{0k}_5 \left[d - \sum_{i=1}^{5} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] I_5 \tag{10}$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1.

Recursions for tensors – Alternative to dimensional recurrences of scalars

Express any $(5, R)$ pentagon by a $(5, R-1)$ pentagon plus $(4, R-1)$ boxes

[Diakonidis,Fleischer,T.Riemann,Tausk: Phys.Lett. **B683** (2010) [21]]

5-point tensor recursion:

$$I_5^{\mu_1\cdots\mu_{R-1}\mu} = I_5^{\mu_1\cdots\mu_{R-1}} Q_0^{\mu} - \sum_{s=1}^{5} I_4^{\mu_1\cdots\mu_{R-1},s} Q_s^{\mu},$$

For $n = 6, 7, 8, \cdots$ things are close but differ a bit; see later.

auxiliary vectors with inverse Gram determinants

$$Q_s^{\mu} = \sum_{i=1}^{5} q_i^{\mu} \frac{\binom{s}{i}_5}{()_5}, \quad s = 0, \ldots, 5$$

For e.g. $R = 3$, again $[1/()_5]^3$ will occur.

Contractions

One may combine now *Tarasov's dimensional recurrence relations* and the *tensor rank recursions* in order to derive especially useful representations.

After that, we will perform contractions with external momenta.
So, the following equations are symbolic:

$$q_{i_1\mu_1}\cdots q_{i_R\mu_R}\ \ I_5^{\mu_1\cdots\mu_R} = \int \frac{d^d k}{i\pi^{d/2}}\ \frac{\prod_{r=1}^{R}(q_{i_r}\cdot k)}{\prod_{j=1}^{5} c_j},$$

$$g_{\mu_1,\mu_2} q_{i_1\mu_3}\cdots q_{i_R\mu_R}\ \ I_5^{\mu_1\cdots\mu_R} \neq \int \frac{k^2 d^d k}{i\pi^{d/2}}\ \frac{\prod_{r=3}^{R}(q_{i_r}\cdot k)}{\prod_{j=1}^{5} c_j}$$

One may arrange a one-loop calculation such that all the one-loop integrals appear only in such contractions.
Important:
The contraction with g_{μ_1,μ_2} is shown here in a symbolic form; in practice we work strictly 4-dimensional with g_{μ_1,μ_2}.

One option was to avoid the appearance of inverse Gram determinants $1/()_5$.

For rank $R = 5$, e.g.:

$$I_5^{\mu\,\nu\,\lambda\rho\sigma} = \sum_{s=1}^{5}\left[\sum_{i,j,k,l,m=1}^{5} q_i^{\mu} q_j^{\nu} q_k^{\lambda} q_l^{\rho} q_m^{\sigma} E_{ijklm}^{s} + \sum_{i,j,k=1}^{5} g^{[\mu\nu} q_i^{\lambda} q_j^{\rho} q_k^{\sigma]} E_{00ijk}^{s} + \sum_{i=1}^{5} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^{s}\right] \quad (11)$$

The tensor coefficients are expressed in terms of integrals $I_{4,i\cdots}^{[d+]^l,s}$, e.g.:

$$E_{ijklm}^{s} = -\frac{1}{\binom{0}{0}_5}\left\{\left[\binom{0l}{sm}_5 n_{ijk} I_{4,ijk}^{[d+]^4,s} + (i \leftrightarrow l) + (j \leftrightarrow l) + (k \leftrightarrow l)\right] + \binom{0s}{0m}_5 n_{ijkl} I_{4,ijkl}^{[d+]^4,s}\right\}.$$

Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $()_4$.
Further, the complete dependence on the indices *i* of the tensor coefficients is contained now in the pre-factors with signed minors.
One can say that **the indices *decouple* from the integrals**.

As an example, we reproduce the 4-point part of $I_{4,ijkl}^{[d+]^4}$:

$$n_{ijkl} I_{4,ijkl}^{[d+]^4} = \frac{\binom{0}{i}}{\binom{0}{0}} \frac{\binom{0}{j}}{\binom{0}{0}} \frac{\binom{0}{k}}{\binom{0}{0}} \frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+]^4}$$
$$+ \frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0j}{0k}\binom{0}{i}\binom{0}{l} + \binom{0i}{0l}\binom{0}{j}\binom{0}{k} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3}$$
$$\times \; d(d+1) I_4^{[d+]^3}$$
$$+ \frac{\binom{0i}{0l}\binom{0j}{0k} + \binom{0j}{0l}\binom{0i}{0k} + \binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+]^2} \quad + \cdots \tag{12}$$

In (12), one has to understand the 4-point integrals to carry the corresponding index s and the signed minors are $\binom{0}{k} \to \binom{0s}{ks}_5$ etc.

- ✓ no scalar 5-point integrals in higher dimensions
- ✓ no inverse Gram det. $()_5$
- ✓ 4-point integrals without indices

† scalar 4-point integrals in higher dimensions: $I_4^{[d+]^2,s}$ etc.
† inverse Gram det. $\binom{0}{0}_5 \equiv ()_4$

Contractions for the 5-point functions with rank $R = 1$

A chord is the momentum shift of an internal line due to external momenta, $D_i = (k - q_i)^2 - m_i^2 + i\epsilon$, and $q_i = (p_1 + p_2 + \cdots + p_i)$, with $q_n = 0$.

The tensor 5-point integral of rank $R = 1$ is ([19], eq. (4.6)):

$$I_5^\mu = -\sum_{i=1}^{5} q_i^\mu I_{5,i}^{[d+]} \tag{13}$$

$$= -\sum_{i=1}^{4} q_i^\mu \sum_{s=1}^{5} \frac{\binom{0i}{0s}_5}{\binom{0}{0}_5} I_4^s \tag{14}$$

This yields, when contracted with a chord,

$$q_{a\mu} I_5^\mu = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left[\sum_{i=1}^{4} (q_a \cdot q_i) \binom{0i}{0s}_5 \right] I_4^s. \tag{15}$$

In fact, the sum over i may be performed explicitly:

$$\Sigma_a^{1,s} \equiv \sum_{i=1}^{4} (q_a \cdot q_i) \binom{0s}{0i}_5 = +\frac{1}{2} \left\{ \binom{s}{0}_5 (Y_{a5} - Y_{55}) + \binom{0}{0}_5 (\delta_{as} - \delta_{5s}) \right\},$$

We get immediately

$$q_{a\mu} I_5^{\mu} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \Sigma_a^{1,s} \quad I_4^s. \tag{16}$$

Contractions for the 5-point functions with rank $R = 2$

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu}\, E_{ij} + g^{\mu\nu} E_{00}, \qquad (17)$$

has the following tensor coefficients free of $1/()_5$:

$$E_{00} = -\sum_{s=1}^{5} \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, \qquad (18)$$

$$E_{ij} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. \qquad (19)$$

Equation (17) yields for the contractions with chords:

$$q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}. \qquad (20)$$

and finally (20) simply reads

$$\begin{aligned} q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} \quad = \quad & \frac{1}{4} \sum_{s=1}^{5} \Bigg\{ \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} \left(\delta_{ab}\delta_{as} + \delta_{5s}\right) + \frac{\binom{s}{s}_5}{\binom{0s}{0s}_5} \Big[\left(\delta_{as} - \delta_{5s}\right) \left(Y_{b5} - Y_{55}\right) \\ & + \left(\delta_{bs} - \delta_{5s}\right) \left(Y_{a5} - Y_{55}\right) + \frac{\binom{s}{0}_5}{\binom{0}{0}_5} \left(Y_{a5} - Y_{55}\right) \left(Y_{b5} - Y_{55}\right) \Big] \Bigg\} I_4^{[d+],s} \\ & + \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\Sigma_b^{1,s}}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \Sigma_a^{2,st} I_3^{st}, \end{aligned}$$

with

$$\Sigma_a^{2,st} \equiv \sum_{i=1}^{4} (q_a \cdot q_i) \begin{pmatrix} 0st \\ 0si \end{pmatrix}_5$$

$$= \frac{1}{2}(1-\delta_{st}) \left\{ \begin{pmatrix} ts \\ 0s \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0s \\ 0s \end{pmatrix}_5 (\delta_{at} - \delta_{5t}) - \begin{pmatrix} 0s \\ 0t \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right\}$$

This has been extended also to higher ranks.
We need at most double sums, e.g.:

$$\Sigma_{ab}^{2,s} \equiv \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} si \\ sj \end{pmatrix}_5$$

$$= \frac{1}{2}(q_a \cdot q_b) \begin{pmatrix} s \\ s \end{pmatrix}_5 - \frac{1}{4}()_5 (\delta_{ab}\delta_{as} + \delta_{5s}), \qquad (21)$$

The sums over signed minors, weighted with scalar products of chords are given in J. Fleischer, T.R., PLB 2011 [10].

Contractions for the 5-point functions with rank $R = 4$

$$I_5^{\mu\nu\lambda\rho} = I_5^{\mu\nu\lambda} \cdot Q_0^{\rho} - \sum_{s=1}^{5} I_4^{\mu\nu\lambda,s} \cdot Q_s^{\rho} . \tag{22}$$

Contracted with chords (differences of external momenta):

$$q_a^{\mu} q_b^{\nu} q_c^{\lambda} q_d^{\rho} \; I_5^{\mu\nu\lambda\rho} = q_a^{\mu} q_b^{\nu} q_c^{\lambda} q_d^{\rho} \; I_5^{\mu\nu\lambda} \; Q_0^{\rho} - C_{5,abcd} \tag{23}$$

Here:

$$Q_{\sigma}^{\mu} \quad = \quad \sum_{i=1}^{5} q_i^{\mu} \frac{\binom{\sigma}{i}_5}{()_5}, \quad \sigma = 0 \cdots 5 \tag{24}$$

The first term $q_a^{\mu} q_b^{\nu} q_c^{\lambda} \; I_5^{\mu\nu\lambda}$ is known, the second term has to be determined:

$$C_{5,abcd} = - \sum_{s=1}^{5} q_{a\mu} q_{b\nu} q_{c\lambda} I_4^{\mu\nu\lambda,s} \frac{1}{2} \left(\delta_{ds} - \delta_{5s} \right) , \tag{25}$$

becomes:

$$
\begin{aligned}
C_{5,abcd} = \frac{1}{16}\Big\{ & G^5 + \delta_{ab}\delta_{ac}\delta_{ad}G^d - I_1^{5abc} - I_1^{5abd} - I_1^{5acd} - I_1^{5bcd} + I_1^{abcd} - J_3^{a5} - J_3^{b5} - J_3^{c5} - J_3^{d5} \\
& + R^{5ab} + R^{5ac} + R^{5bc} + R^{5da} + R^{5db} + R^{5dc} + \delta_{bc}\delta_{bd}\left(J_3^{ad} - J_3^{5d}\right) + \delta_{ac}\delta_{ad}\left(J_3^{bd} - J_3^{5d}\right) \\
& + \delta_{ab}\delta_{ad}\left(J_3^{cd} - J_3^{5d}\right) + \delta_{ab}\delta_{ac}\left(J_3^{dc} - J_3^{5c}\right) + \delta_{ab}\delta_{cd}\tilde{J}_3^{db} + \delta_{ad}\delta_{bc}\tilde{J}_3^{dc} + \delta_{ac}\delta_{bd}\tilde{J}_3^{dc} \\
& + \delta_{ab}\left(\tilde{J}_3^{5b} - R^{b5c} - R^{bd5} + R^{bdc}\right) + \delta_{ac}\left(\tilde{J}_3^{5c} - R^{c5b} - R^{cd5} + R^{cdb}\right) \\
& + \delta_{ad}\left(\tilde{J}_3^{d5} - R^{d5b} - R^{d5c} + R^{dbc}\right) + \delta_{bc}\left(\tilde{J}_3^{5c} - R^{c5a} - R^{cd5} + R^{cda}\right) \\
& + \delta_{bd}\left(\tilde{J}_3^{d5} - R^{d5a} - R^{d5c} + R^{dac}\right) + \delta_{cd}\left(\tilde{J}_3^{d5} - R^{d5a} - R^{d5b} + R^{dab}\right) \\
& + \delta_{ab}\delta_{ad}J_4^d Y_c + \delta_{ac}\delta_{ad}J_4^d Y_b + \delta_{bc}\delta_{bd}J_4^d Y_a + \delta_{ab}\left(R^{bd} - R^{b5}\right)Y_c + \delta_{ac}\left(R^{cd} - R^{c5}\right)Y_b \\
& + \delta_{ad}\left(R^{dc} - R^{d5}\right)Y_b + \delta_{ad}\left(R^{db} - R^{d5}\right)Y_c + \delta_{bc}\left(R^{cd} - R^{c5}\right)Y_a + \delta_{bd}\left(R^{dc} - R^{d5}\right)Y_a \\
& + \delta_{bd}\left(R^{da} - R^{d5}\right)Y_c + \delta_{cd}\left(R^{da} - R^{d5}\right)Y_b + \delta_{cd}\left(R^{db} - R^{d5}\right)Y_a + \left(I_4^d - I_4^5\right)Y_aY_bY_c \\
& + \left(I_3^{cd} - I_3^{5c} - I_3^{5d} + R^5 + \delta_{cd}R^d\right)Y_aY_b + \left(I_3^{bd} - I_3^{5b} - I_3^{5d} + R^5 + \delta_{bd}R^d\right)Y_aY_c \\
& + \left(I_3^{ad} - I_3^{5a} - I_3^{5d} + R^5 + \delta_{ad}R^d\right)Y_bY_c + \left(I_2^{bcd} - I_2^{5bc} - I_2^{5bd} - I_2^{5cd} - J_4^5 + R^{5b} + R^{5c} + R^{5d}\right)Y_a \\
& + \left(I_2^{acd} - I_2^{5ac} - I_2^{5ad} - I_2^{5cd} - J_4^5 + R^{5a} + R^{5c} + R^{5d}\right)Y_b \\
& + \left(I_2^{abd} - I_2^{5ab} - I_2^{5ad} - I_2^{5bd} - J_4^5 + R^{5a} + R^{5b} + R^{5d}\right)Y_c\Big\}
\end{aligned} \tag{26}
$$

where we have introduced:

$$J_3^{st} \equiv \frac{1}{\binom{st}{st}_5} \left\{ -\binom{s}{s}_5 I_3^{[d+],st} + \binom{ts}{0s}_5 R^{ts} - \sum_{u=1}^{5} \binom{ts}{us}_5 R^{tsu} \right\}, \quad (27)$$

$$\tilde{J}_3^{st} \equiv \frac{1}{\binom{st}{st}_5} \left\{ \binom{s}{t}_5 I_3^{[d+],st} + \binom{st}{0t}_5 R^{ts} - \sum_{u=1}^{5} \binom{st}{ut}_5 R^{tsu} \right\}, \quad (28)$$

$$G^s \equiv \frac{1}{\binom{s}{s}_5} \left\{ -2\binom{}{}_5 R^{[d+],s} + \binom{s}{0}_5 J_4^s - \sum_{t=1}^{5} \binom{s}{t}_5 J_3^{ts} \right\}. \quad (29)$$

J_4^s and $R^{[d+],s}$ are given in Eqs. (2.24) and (2.44) of of [19] respectively.

We assume throughout that $q_5 = 0$, where $q_1, \ldots, q_5$ are chords – differences of external momenta.

Further abbreviations (see (2.24, 2.49, 2.9, 2.17, 2.34, 2.41) of [19]) :

$$J_4^s \equiv \frac{1}{\binom{s}{s}_5}\left\{-()_5 I_4^{[d+],s} + \binom{s}{0}_5 R^s - \sum_{t=1}^{5}\binom{s}{t}_5 R^{st}\right\} \tag{30}$$

$$= \frac{-1}{\binom{0s}{0s}_5}\left\{()_5\,(d-2)(d-1)I_4^{[d+]^2,s} - \binom{0}{0}_5 I_4^{[d+],s} + \sum_{t=1}^{5}\binom{t}{0}_5 (d-2)I_3^{[d+],st} + \sum_{t=1}^{5}\binom{0s}{0t}_5 R^{st}\right\}$$

$$R^{[d+],s} = \frac{1}{\binom{s}{s}_5}\left[\binom{s}{0}_5 I_4^{[d+],s} - \sum_{t=1}^{5}\binom{s}{t}_5 I_3^{[d+],st}\right] = \frac{1}{\binom{0s}{0s}_5}\left[\binom{s}{0}_5 (d-1)I_4^{[d+]^2,s} - \sum_{t=1}^{5}\binom{0s}{0t}_5 I_3^{[d+],st}\right] \tag{31}$$

$$R^s \equiv \frac{1}{\binom{s}{s}_5}\left[\binom{s}{0}_5 I_4^s - \sum_{t=1}^{5}\binom{s}{t}_5 I_3^{st}\right] = \frac{1}{\binom{0s}{0s}_5}\left[\binom{s}{0}_5 I_4^{[d+],s} - \sum_{t=1}^{5}\binom{0s}{0t}_5 I_3^{st}\right]. \tag{32}$$

$$R^{st} \equiv \frac{1}{\binom{st}{st}_5}\left[\binom{st}{0t}_5 I_3^{st} - \sum_{u=1}^{5}\binom{st}{ut}_5 I_2^{stu}\right] = \frac{1}{\binom{0st}{0st}_5}\left[\binom{st}{0t}_5 (d-2)I_3^{[d+],st} - \sum_{u=1}^{5}\binom{0st}{0ut}_5 I_2^{stu}\right]. \tag{33}$$

$$R^{tsu} = \frac{1}{\binom{stu}{stu}_5}\left[\binom{tsu}{0su}_5 I_2^{stu} - \sum_{v=1}^{5}\binom{tsu}{vsu}_5 I_1^{stuv}\right]$$

$$= \frac{1}{\binom{0stu}{0stu}_5}\left[\binom{tsu}{0su}_5 (d-1)I_2^{[d+],stu} - \sum_{v=1}^{5}\binom{0tsu}{0vsu}_5 I_1^{stuv}\right]. \tag{34}$$

$$Y_a = Y_{a5} - Y_{55}, \quad Y_{ab} = -(q_a - q_b)^2 + m_a^2 + m_b^2 \tag{35}$$

Some numerics for 5-point function with rank $R = 4$

We compare with LoopTools/FF/OneLOop

```
  the kinematics:
p1s =  1.1163688400000000E-002  p2s =  2.6109999999999998E-007  p3s =  0.0000000000000000
p4s =  2.6109999999999998E-007  p5s =  1.1163688400000000E-002
s12 = -0.70858278190000001      s23 = -1.5343299000000002E-003  s34 = -0.12851860429999998
s45 = -0.61023937949999996      s15 =  0.92668942420000000
m1s =  1.1163688361676107E-002  m2s =  0.0000000000000000       m3s =  2.6112003932088364E-00
m4s =  2.6112003932088364E-007  m5s =  0.0000000000000000

OneLOop-3.3.1
 for the evaluation of 1-loop scalar 1-, 2-, 3- and 4-point functions
van Hameren arXiv:1007.4716 and van Hameren, Papadopoulos, Pittau arXiv:0903.4665

 the R=4 contractions, a,b,c,d=3,3,3,3  ( -48094.1074 54542318     , -47802.08746 5035322 )
      LoopTools                         ( -48094.1074 65           , -47802.08746 05      )

 the R=4 contractions, a,b,c,d=3,3,3,4  ( -18463.1204 24842149     , -23446.4704 12257226 )
               LoopTools                ( -18463.1204 31           , -23446.4704 09       )

 the R=4 contractions, a,b,c,d=3,3,3,5  (  0.0000000000000000     ,  0.0000000000000000 )
               LoopTools                (  0.0000000000000000     ,  0.0000000000000000 )

compared to: LoopTools/FF, Hahn, arXiv:hep-ph/9807565 and van Oldenborgh, CPC66(1991)
```

Summary

- Explicit Analytical, recursive treatment of **heptagon, hexagon and pentagon tensor integrals** of rank R in terms of pentagons and boxes of rank $R-1$
- Systematic derivation of expressions which are explicitely free of inverse Gram determinants $()_5$ until pentagons of rank $R=5$
- Numerical tensor reduction package PJFry for C, C++, Mathematica, Fortran
- Numerical packages for contracted tensor integrals OLEC and CONTRACTIONS for C++ and Fortran under development

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