Scalar one-loop integrals as meromorphic functions of space-time dimension d

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Why one-loop Feynman integrals? And why in $D = 4 + 2n - 2\epsilon$ dimensions? I

I began in 1980 to calculate Feynman integrals, and after several proceedings contributions, published an article, Mann, Riemann, 1983 [1]: "Effective Flavor Changing Weak Neutral Current In The Standard Theory And Z Boson Decay"

Basics

The seminal papers on 1-loop Feynman integrals:
't Hooft, Veltman, 1978 [2]: "Scalar oneloop integrals"
Passarino, Veltman, 1978 [3]: "One Loop Corrections for e^+e^- Annihilation into $\mu^+\mu^-$ in the Weinberg Model"

Interest in "modern" developments for the calculation of 1-loop integrals from basically two sides

For many-particle calculations, there appear inverse Gram determinants from tensor reductions in the answers.

These $1/G(p_i)$ may diverge, because Gram dets can exactly vanish: $G(p_i) \equiv 0$.

One may perform tensor reductions so that no inverse Grams appear, but one has to buy 1-loop integrals in higher dimensions, $D = 4 + 2n - 2\epsilon$.

Why one-loop Feynman integrals? And why in $D = 4 + 2n - 2\epsilon$ dimensions? II

Key references for tensor reductions etc., , I give here no complete list

Davydychev, 1991 [4]: "A Simple formula for reducing Feynman diagrams to scalar integrals"
This paper explains how to write tensor integrals as scalar integrals with higher indices and in higher dimensions. Lowering of indices and/or dimensions by recursive reductions were introduced:
Tkachov,Chetyrkin 1981 [5, 6]: Integration-by-parts identities
Tarasov 1996 [7], Fleischer, Jegerlehner, Tarasov 1999 [8]: plus dimensional shifts (downwards), they introduce the inverse Gram dets $1/G(p_i)$
Fleischer, Riemann 2010–2013 [9, 10] and other papers: Ensure that inverse Gram dets $1/G(p_i)$ do not destabilize (Gram dets are avoided, or integrals are expanded) and that all indices are equal one:

Higher-order loop calculations need h.o. contributions from ϵ-expansions of 1-loops: $1/(d-4) = -1/(2\epsilon)$ and $\Gamma(\epsilon) = a/\epsilon + c + \epsilon + \cdots$

A Seminal paper was on ϵ-terms of 1-loop functions:
Nierste, Müller, Böhm, 1992 [11]: "Two loop relevant parts of D-dimensional massive scalar one loop integrals"
This was generalized in another 2 seminal papers:
Tarasov, 2000 [12] and Fleischer, Jegerlehner, Tarasov, 2003 [13]: "A New hypergeometric representation of one loop scalar integrals in d dimensions"

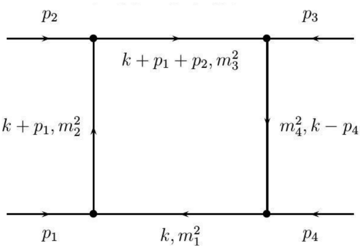
Why one-loop Feynman integrals? And why in $D = 4 + 2n - 2\epsilon$ dimensions? III

I was wondering if the results of Fleischer/Jegerlehner/Tarasov (2003) are sufficiently general for practical, black-box applications, and saw a need of creating a software solution in terms of contemporary mathematics.

So we decided to study the issue from scratch in 2 steps:

1st step: Re-derive analytical expressions for
scalar one-loop integrals as meromorphic functions of **arbitrary** space-time dimension D
Approve or improve the results of Tarasov et al.

- 2-point functions: Gauss hypergeometric functions ${}_2F_1$ [14]
  3-point functions: plus Kamp'e de F'eriet functions F_1; there are the Appell functions $F_1, \ldots F_4$ [15]
  4-point functions: plus Lauricella-Saran functions F_S [16]
- **2nd step:** Derive the Laurent expansions around the singular points of these functions.
- This talk:
  Self-energies and vertices
- We have preliminary results also for boxes but want to perform more numerical checks.



$$J_N \equiv J_N(d;\{p_ip_j\},\{m_i^2\}) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_N^{\nu_N}} \tag{1}$$

with

$$D_i = \frac{1}{(k+q_i)^2 - m_i^2 + i\epsilon}. \tag{2}$$

$$\nu_i = 1, \quad \sum_{i=1}^{n} p_i = 0 \tag{3}$$

$$J_n = (-1)^n \Gamma(n-d/2) \int_0^1 \prod_{j=1}^{n} dx_j \delta\left(1 - \sum_{i=1}^{n} x_i\right) \frac{1}{F_n(x)^{n-d/2}} \tag{4}$$

Here, the F-function is the second Symanzik polynomial.

It is derived from the propagators (2),

$$M^2 = x_1 D_1 + \cdots + x_N D_N = k^2 - 2Qk + J. \tag{5}$$

Using $\delta(1 - \sum x_i)$ under the integral in order to transform linear terms in x into quadratic ones, we may obtain:

$$F_n(x) = -(\sum_i x_i)\, J + Q^2 = \frac{1}{2} \sum_{i,j} x_i Y_{ij} x_j - i\epsilon, \tag{6}$$

The Y_{ij} are elements of the Cayley matrix, introduced for a systematic study of one-loop n-point Feynman integrals e.g. in [17]

$$Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2. \tag{7}$$

One-point function, or tadpole

$$J_1(d;m^2) \quad = \quad \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2-m^2+i\epsilon} \quad = \quad -\frac{\Gamma(1-d/2)}{(m^2-i\epsilon)^{1-d/2}}. \qquad (8)$$

The operator $\mathbf{k}^-$ ...

...will reduce an n-point Feynman integral J_n to an $(n-1)$-point integral J_{n-1} by shrinking the propagator $1/D_k$

$$\mathbf{k}^- \; J_n \quad = \quad \mathbf{k}^- \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{\prod_{j=1}^n D_j} \quad = \quad \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{\prod_{j\neq k,j=1}^n D_j}. \qquad (9)$$

Mellin-Barnes representation

$$\frac{1}{(1+z)^\lambda} \quad = \quad \frac{1}{2\pi i} \int\limits_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(\lambda+s)}{\Gamma(\lambda)} \, z^s \quad = \quad {}_2F_1\left[\begin{matrix} \lambda, b\,; \\ b\,; \end{matrix} \; -z\right]. \qquad (10)$$

It is valid if $|\mathrm{Arg}(z)| < \pi$ and the integration contour has to be chosen such that the poles of $\Gamma(-s)$ and $\Gamma(\lambda+s)$ are well-separated. The right hand side of (10) is identified as Gauss' hypergeometric function. For more details see [18]).

F-function and Gram, Cayley, and modified Cayley determinants

Introduced by Melrose [17]. The Cayley determinant $\lambda_{12\ldots N}$ is composed of the $Y_{ij} = m_i^2 + m_j^2 - (q_i - q_j)^2$ introduced in (7), and its determinant is:

$$\lambda_n \equiv \lambda_{12\ldots n} \quad = \quad \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}. \tag{11}$$

The modified Cayley determinant is

$$()_n \quad = \quad \begin{vmatrix} 0 & 1 & \ldots & 1 & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}. \tag{12}$$

Here, the additional definitions $Y_{00} = 0, \;\; Y_{0j} = Y_{j0} = 1, \;\; i,j = 1,\ldots,n$ are introduced.

We also define the $(n-1) \times (n-1)$ dimensional Gram determinant $g_n \equiv g_{12\cdots n}$,

$$G_n \equiv G_{12\cdots n} = - \begin{vmatrix} (q_1-q_n)^2 & (q_1-q_n)(q_2-q_n) & \ldots & (q_1-q_n)(q_{n-1}-q_n) \\ (q_1-q_n)(q_2-q_n) & (q_2-q_n)^2 & \ldots & (q_2-q_n)(q_{n-1}-q_n) \\ \vdots & \vdots & \ddots & \vdots \\ (q_1-q_n)(q_{n-1}-q_n) & (q_2-q_n)(q_{n-1}-q_n) & \ldots & (q_{n-1}-q_n)^2 \end{vmatrix} . \quad (13)$$

The determinants are independent of a common shifting of the momenta q_i.
Further, the Gram det G_n and the modified Cayley determinant $()_n$ are **independent of the propagator masses**.
For the Gram determinant this is evident, and the following relation between both determinants holds, for arbitrary q_i:

$$()_n \quad = \quad g_n \equiv -2^{n-1} G_n. \qquad (14)$$

Co-factors of the Cayley matrix

One further notation will be introduced, namely that of co-factors of the Cayley matrix. Also called signed minors in e.g. [17, 19]):

$$\begin{pmatrix} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{pmatrix}_n . \tag{15}$$

The signed minors are determinants, labeled by those rows $j_1, j_2, \cdots j_m$ and columns $k_1, k_2, \cdots k_m$ which have been discarded from the definition of the Cayley determinant $()_n$, with a sign convention.

$$\text{sign} \begin{pmatrix} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{pmatrix}_n = (-1)^{j_1+j_2+\cdots+j_m+k_1+k_2+\cdots+k_m} \times \texttt{Signature}[j_1, j_2, \cdots j_m] \times \texttt{Signature}[k_1, k_2, \cdots k_m] \tag{16}$$

Here, `Signature` (defined like the Mathematica command) gives the sign of permutations needed to place the indices in increasing order.

Cayley matrix, by definition:

$$\lambda_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n . \tag{17}$$

Further, it is (see [8]):

$$-\frac{1}{2}\,\partial_i \lambda_n \equiv -\frac{1}{2}\,\frac{\partial \lambda_n}{\partial m_i^2} = \begin{pmatrix} 0 \\ i \end{pmatrix}_n . \tag{18}$$

Rewriting the F-function further, exploring the $x_n = 1 - \sum x_i$ ...

The elimination of one of the x_i creates linear terms in $F(x)$.

$$F_n(x) = x^T G_n x + 2H_n^T x + K_n. \tag{19}$$

The $F_n(x)$ may be cast by shifts $x \to (x - y)$ into the form

$$F_n(x) = (x-y)^T G_n (x-y) + r_n - i\varepsilon = \Lambda_n(x) + r_n - i\varepsilon = \Lambda_n(x) + R_n, \tag{20}$$

with

$$\Lambda_n(x) = (x-y)^T G_n (x-y), \tag{21}$$

and

$$\begin{aligned} r_n &= K_n - H_n^T G_n^{-1} H_n \\ &= -\frac{\lambda_n}{g_n} \\ &= -\frac{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n}{()_n}. \end{aligned} \tag{22}$$

The linear shifts y_i

The $(n-1)$ components y_i of the vector y appearing here in $F_n(x)$ are:

$$y_i = -\left(G_n^{-1} K_n\right)_i, \quad i \neq n \tag{23}$$

The following relations are also valid:

$$y_i = \frac{\partial r_n}{\partial m_i^2} = -\frac{1}{g_n}\frac{\partial \lambda_n}{\partial m_i^2} = -\frac{\partial_i \lambda_n}{g_n} = \frac{2}{g_n}\begin{pmatrix} 0 \\ i \end{pmatrix}_n, \quad i = 1 \cdots n. \tag{24}$$

The auxiliary condition $\sum_i^n y_i = 1$ is fulfilled.

We see that the notations for the F-function are finally independent of the choice of the variable which was eliminated by use of the δ-function in the integrand of (4). The inhomogeneity R_n is the only variable carrying the causal $i\epsilon$-prescription, while e.g. $\Lambda(x)$ and the y_i are by definition real quantities.

The recursion relation for J_n I

One may use the Mellin-Barnes relation (10) in order to decompose the integrand of J_n given in (4) as follows:

$$\frac{1}{[F(x)]^{n-\frac{d}{2}}} \equiv \frac{1}{[\Lambda_n(x)+R_n]^{n-\frac{d}{2}}} \equiv \frac{R_n^{-(n-\frac{d}{2})}}{[1+\frac{\Lambda_n(x)}{R_n}]^{n-\frac{d}{2}}}$$
$$= \frac{R_n^{-(n-\frac{d}{2})}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(n-\frac{d}{2}+s)}{\Gamma(n-\frac{d}{2})} \left[\frac{\Lambda_n(x)}{R_n}\right]^s, \qquad (25)$$

for $|\mathrm{Arg}(\Lambda_n/R_n)| < \pi$. The condition always applies. Further, the integration path in the complex s-plane separates the poles of $\Gamma(-s)$ and $\Gamma(n-\frac{d}{2}+s)$. As a result of (25), the Feynman parameter integral of J_n becomes homogeneous:

$$K_n = \prod_{j=1}^{n-1} \int_0^{1-\sum_{i=j+1}^{n-1} x_i} dx_j \left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \int dS_{n-1} \left[\frac{\Lambda_n(x)}{R_n}\right]^s. \qquad (26)$$

The recursion relation for J_n II

In order to solve this integral, we consider the differential operator $\hat{P}_n$ [20, 21],

$$\hat{P}_n \left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \sum_{i=1}^{n-1} \frac{1}{2}(x_i - y_i)\frac{\partial}{\partial x_i} \left[\frac{\Lambda_n(x)}{R_n}\right]^s = s \left[\frac{\Lambda_n(x)}{R_n}\right]^s . \tag{27}$$

This eigenvalue relation allows to introduce the operator $\hat{P}_n$ into the integrand of (26):

$$K_n = \frac{1}{s}\int dS_{n-1}\, \hat{P}_n \left[\frac{\Lambda_n(x)}{R_n}\right]^s = \frac{1}{2s}\sum_{i=1}^{n-1}\prod_{k=1}^{n-1}\int_0^{u_k} dx'_k\, (x_i - y_i)\frac{\partial}{\partial x_i}\left[\frac{\Lambda_n(x)}{R_n}\right]^s . \tag{28}$$

After a series of manipulations in order to perform one of the x-integrations – by partial integration, eating the corresponding differential – one arrives at:

$$J_n = \frac{(-1)^n}{2\pi i}\int_{-i\infty}^{+i\infty} ds\, \frac{\Gamma(-s)\,\Gamma(n-\frac{d}{2}+s)\Gamma(s+1)}{2\,\Gamma(s+2)} \left(\frac{1}{R_n}\right)^{n-\frac{d}{2}}$$

$$\times \sum_{i=1}^{n}\left\{\left(\frac{\partial r_n}{\partial m_i^2}\right)\int dS_{n-2}^{(i)} \left[\frac{F_{n-1}^{(i)}}{R_n} - 1\right]^s \right. \tag{29}$$

The recursion relation for J_n III

We stress again that only the R_n carries an $i\epsilon$. Now it is important to eliminate the term (-1) from the combination $(F_{n-1}^{(i)}/R_n - 1)^s$ under the Mellin-Barnes integral over s, because then we arrive at a sum over the n different $(n-1)$-point functions arising from skipping a propagator from the original integral. In fact, this may be arranged using the following relation for $(-z) = F/R - 1$ [22]:

$$\int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(a+s)\,\Gamma(b+s)}{\Gamma(c+s)} (-z)^s \tag{30}$$

$$= \int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(a+b-c-s)\Gamma(c-a+s)\Gamma(c-b+s)}{\Gamma(c-a)\Gamma(c-b)} \, (1-z)^{c-a-b+s},$$

provided that $|\mathrm{Arg}(-z)| < 2\pi$.
We arrive at the following recursion relation:

The recursion relation for J_n IV

$$J_n(d,\{q_i,m_i^2\}) = \frac{-1}{2\pi i}\int_{-i\infty}^{+i\infty} ds\, \frac{\Gamma(-s)\,\Gamma(\frac{d-n+1}{2}+s)\Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} R_n^{-s} \times \sum_{k=1}^{n}\left(\frac{1}{r_n}\frac{\partial r_n}{\partial m_k^2}\right)\mathbf{k}^- J_n(d+2s;\{q_i,m_i^2\}). \quad (31)$$

The cases $G_n = 0$ and $\lambda_n = r_n = 0$ prevent the use of the Mellin-Barnes transformation. They are simpler than what we have to do here. Details are given elsewhere.

The 2-point function

From our recursion relation (36), taken at $n = 2$ and using the expression (8) with $d \to d + 2s$ for the one-point functions under the integral, one gets the following representation:

$$J_2(D; q_1, m_1^2, q_2, m_2^2) = \frac{e^{\epsilon\gamma_E}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma\left(\frac{D-1}{2}+s\right)\,\Gamma(s+1)}{2\,\Gamma\left(\frac{D-1}{2}\right)} R_2^s \times \left[\frac{1}{r_2} \frac{\partial r_2}{\partial m_2^2} \frac{\Gamma\left(1 - \frac{D+2s}{2}\right)}{(m_1^2)^{1-\frac{D+2s}{2}}} + (1 \leftrightarrow 2) \right]. \tag{32}$$

One may close the integration contour of the MB-integral in (36) to the right, apply the Cauchy theorem and collect the residua originating from two series of zeros of arguments of Γ-functions at $s = m$ and $s = m - d/2 - 1$ for $m \in \mathbb{N}$.

The first series stems from the MB-integration kernel, the other one from the dimensionally shifted 1-point functions.

And then summing up in terms of Gauss' hypergeometric functions.

We get eqn. (53) of [13]:

$$J_2(d;q_1,m_1^2,q_2,m_2^2) \quad = \quad J_2^{(1)} + J_2^{(2)}, \qquad (33)$$

and

$$J_2^{(1)} \quad = \quad -\frac{\Gamma\left(2-\frac{d}{2}\right)\;\Gamma\left(\frac{d}{2}-1\right)}{2\,\Gamma\left(\frac{d}{2}\right)} \left\{ \left(\frac{1}{r_{12}}\frac{\partial r_{12}}{\partial m_2^2}\right) \frac{(m_1^2)^{\frac{d}{2}-1}}{\sqrt{1-\frac{m_1^2}{R_{12}}}}\; {}_2F_1\left[\begin{matrix}\frac{d}{2}-1,\frac{1}{2}; \\ \frac{d}{2};\end{matrix}\;\frac{m_1^2}{R_{12}}\right] + (1\leftrightarrow 2)\right.$$

$$J_2^{(2)} \quad = \quad \frac{\sqrt{\pi}\;\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)}{2\,\Gamma(\frac{d-1}{2})} \frac{(R_{12})^{\frac{d}{2}-1}}{\lambda_{12}} \left[\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}} + \frac{\partial_1\lambda_{12}}{\sqrt{1-\frac{m_2^2}{R_{12}}}}\right]. \qquad ($$

The representation (33) is valid for $\left|\frac{m_1^2}{r_{12}}\right| < 1$, $\left|\frac{m_2^2}{r_{12}}\right| < 1$ and $\mathcal{R}\mathrm{e}(\frac{d-2}{2}) > 0$. It is in agreement with Eqn. (53) of [13].

The 3-point function I

According to the master formula (36), we can write the massive 3-point function as a sum of three terms:

$$J_3 \quad = \quad J_{123} + J_{231} + J_{312}, \tag{35}$$

using the representation for e.g. J_{123}

$$J_{123}(d, \{q_i, m_i^2\}) \quad = \quad -\frac{e^{\epsilon\gamma_E}}{2\pi i} \int\limits_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(\frac{d-2+2s}{2})\Gamma(s+1)}{2\,\Gamma(\frac{d-2}{2})} R_3^{-s} \times \frac{1}{r_3}\frac{\partial r_3}{\partial m_3^2}\, J_2(d+2s; q_1, m_1^2, q_2, m_2^2). \tag{36}$$

The 3-point function II

Here, $J_2(d+2s; q_1, m_1^2, q_2, m_2^2)$ is given by (33), taken at $d+2s$ dimensions. By performing the Mellin-Barnes integrals, one gets three terms, each consisting of eight series, from taking the residues by closing the integration contours to the right; one of the three terms is:

$$
\begin{aligned}
J_{123} \quad = \quad & \Gamma\left(2-\frac{d}{2}\right) R_{123}^{\frac{d}{2}-2} \times b_{123} \\
& - \frac{\sqrt{\pi}\,\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d-1}{2}\right)} \; \frac{\partial_3\lambda_{123}}{\lambda_{123}} \; \frac{R_{12}^{\frac{d}{2}-1}}{4\lambda_{12}} \left[\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}} + \frac{\partial_1\lambda_{12}}{\sqrt{1-\frac{m_2^2}{R_{12}}}}\right] \\
& \times \, {}_2F_1\left[\begin{matrix} \frac{d-2}{2}, 1 ; \\ \frac{d-1}{2} ; \end{matrix} \; \frac{R_{12}}{R_{123}}\right] + \frac{2}{d-2}\Gamma\left(2-\frac{d}{2}\right) \frac{\partial_3\lambda_{123}}{\lambda_{123}} \qquad (37) \\
& \qquad \times \left[\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}} \frac{(m_1^2)^{\frac{d}{2}-1}}{4\lambda_{12}} F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{m_1^2}{R_{123}}, \frac{m_1^2}{R_{12}}\right) + (1 \leftrightarrow 2)\right],
\end{aligned}
$$

The 3-point function III

and

$$b_{123} = -\frac{1}{2g_{12}} \frac{\partial_3 \lambda_{123}}{\lambda_{123}} \left(\frac{\partial_2 \lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}} + \frac{\partial_1 \lambda_{12}}{\sqrt{1-\frac{m_2^2}{R_{12}}}} \right) {}_2F_1\left[\begin{matrix} 1, 1\,; \\ \frac{3}{2}\,; \end{matrix} \frac{R_{12}}{R_{123}} \right] \tag{38}$$
$$- \frac{\partial_3 \lambda_{123}}{\lambda_{123}} \left\{ \frac{\partial_2 \lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}} \frac{m_1^2}{4\lambda_{12}} F_1\left(1; 1, \frac{1}{2}; 2; \frac{m_1^2}{R_{123}}, \frac{m_1^2}{R_{12}}\right) + (1 \leftrightarrow 2) \right\},$$

where $\partial_i \lambda_{j\ldots}$ is defined in (24). The representation (35) is valid for $\mathrm{Re}\left(\frac{d-2}{2}\right) > 0$. The conditions $\left|\frac{m_i^2}{R_{ij}}\right| < 1$, $\left|\frac{R_{ij}}{R_{ijk}}\right| < 1$ had to be met during the derivation. The result may be analytically continued in a straightforward way, however, in the complete complex domain.
The functions ${}_2F_1$ and F_1 of the b_{ijk}-terms are met by setting $d = 4$ in the corresponding functions J_{ijk} of the general J_3.

For the 3-point function, we look at the expression $J_{123} + J_{231} + J_{312}$.

We should agree with eqn. (74) to (76) of Tarasov 2003.

Our terms with d-dimensional F_1 and ${}_2F_1$ do agree exactly, but $b_{123} + b_{231} + b_{312}$ looks quite different.

Tarasov 2003 [13], eqns. (73) and (75)

Under the kinematic conditions that:

$$G_3 < 0, \quad \frac{m_i^2}{r_3} > 1, \quad p_{ij}^2 < 0: \quad b_3 \neq 0 \tag{39}$$

the “b”-term of Tarasov 2003 becomes:

$$J_3(b_3) \quad = \quad \frac{\Gamma(2-d/2)}{\lambda_3} \left(2^{3/2}\, \pi\, \sqrt{-G_3}\;\; {R_3}^{d/2-1}\right) \tag{40}$$

Otherwise:

$$J_3(b_3) = b_3 = 0. \tag{41}$$

Numerics for 3-point functions, table 1

$[p_i^2]$, $[m_i^2]$	[+100, +200, +300], [10, 20, 30]	
G_{123} λ_{123} m_i^2/r_{123} m_i^2/r_{12} m_i^2/r_{23} m_i^2/r_{31} $\sum J$-terms	–160000 –8860000 –0.180587, –0.361174, –0.541761 –0.97561, –1.95122, –2.92683 –0.39801, –0.79602, –1.19403 –0.180723, –0.361446, –0.542169 (0.019223879 – 0.007987267 I)	
$\sum b_3$-terms J_3(TR)	0 (0.019223879 – 0.007987267 I)	
b_3-term $b_3 + \sum J$-terms	(–0.089171509 + 0.069788641 I) (–0.012307377 – 0.009301346 I)	+ (0.022214414)/eps
J_3(OT)	$\sum J$-terms, b_3-term $\to$ 0, OK	
MB suite		
(-1)*fiesta3	-(0.012307 + 0.009301 I)	+ (8*10-6 + 0.00001 I) pm4)
LoopTools/FF, ϵ^0	0.0192238790286244077-0.00798726725497102795 i	

Table 1: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. Red input quantities suggest that, according to eq. (73) in Tarasov2003 [13], one has to set $b_3 = 0$. Although b_3 of [13] deviates from our vanishing value, it has to be set to zero, so that the results of both calculations for J_3 agree for this case.

Numerics for 3-point functions, table 2

$[p_i^2]$, $[m_i^2]$	[-100, +200, -300], [10, 20, 30]	
G_{123}	480000	
λ_3	-19300000	
m_i^2/r_3	0.248705, 0.497409, 0.746114	
m_i^2/r_{12}	0.248447, 0.496894, 0.745342	
m_i^2/r_{23}	-0.39801, -0.79602, -1.19403	
m_i^2/r_{31}	0.104895, 0.20979, 0.314685	
$\sum J$-terms	(-0.012307377 - 0.056679689 I)	+ (+ 0.012825498 I)/eps
$\sum b_3$-terms	(+ 0.047378343 I)	- (+ 0.012825498 I)/eps
J_3(TR)	(-0.012307377 - 0.009301346 I)	
b_3-term	(+ 0.047378343 I)	- (+ 0.012825498 I)/eps
$b_3+\sum J$-terms	(-0.012307377 - 0.009301346 I)	
J_3(OT)	$\sum J$-terms, b_3-term$\to$0, gets wrong	
MB suite		
(-1)*fiesta3	-(0.012307 + 0.009301 I)	+ (8*10-6 + 0.00001 I) pm4)
LoopTools/FF, ϵ^0	-0.0123073773677820630 - 0.0093013461700863289 i	

Table 2: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. Red input quantities suggest that, according to eq. (73) in Tarasov2003 [13], one has to set $b_3 = 0$. Further, we have set in the numerics for eq. (75) of Tarasov2003 [13] that Sqrt[-g123 + I*epsil], what looks counter-intuitive for a "momentum"-like function.

Numerics for 3-point functions, table 3

p_i^2	–100,–200,–300	
m_i^2	10,20,30	
G_{123}	–160000	
λ_{123}	15260000	
m_i^2/r_{123}	0.104849, 0.209699, 0.314548	
m_i^2/r_{12}	0.248447, 0.496894, 0.745342	
m_i^2/r_{23}	0.133111, 0.266223, 0.399334	
m_i^2/r_{31}	0.104895, 0.20979, 0.314685	
$\sum J$-terms	(0.0933877 – 0 I)	– (0.0222144 – 0 I)/eps
$\sum b$-terms	-0.101249	+ 0.0222144/eps
J_3(TR)	(–0.00786155 – 0 I)	
b_3	(-0.101249 + 0 I)	+ (0.0222144 + 0 I)/eps
b_3+J-terms	(–0.007861546 + 0 I)	
J_3(OT)	b_3+J-terms $\rightarrow$ OK	
MB suite	–0.007862014, 5.002549159*10-6, 0	
(-1)*fiesta3	–(0.007862)	+ (6*10-6 + 6*10-6 I pm10)
LoopTools/FF, ϵ^0	–0.00786154613229082290	

Table 3: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$.

Numerics for 3-point functions, table 4

p_i^2 m_i^2	+100, –200, +300 10, 20, 30	
G_{123} λ_{123} m_i^2/r_{123} m_i^2/r_{12} m_i^2/r_{23} m_i^2/r_{31} $\sum J$-terms	480000 4900000 –0.979592, –1.95918, –2.93878 –0.97561, –1.95122, –2.92683 0.133111, 0.266223, 0.399334 –0.180723, –0.361446, –0.542169 (0.006243624 - 0.018272524 I)	
$\sum b_3$-terms J_3(TR)	0 (0.006243624 - 0.018272524 I)	
b_3-term b_3+$\sum J$-terms	(0.040292491 + 0.029796253 I) (-0.012307377 - 0.009301346 I)	+ (- 0.012825498 I)/eps + (4*-18 - 6*-18 I)/eps
J_3(OT)	$\sum J$-terms, b_3-term→0, OK	
MB suite		
(-1)*fiesta3	-(-0.006322 + 0.014701 I)	+ (0.000012 + 0.000014 I) pm
LoopTools/FF, ϵ^0	0.00624362477277410 - 0.01827252404872805 i	

Table 4: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. Red input quantities suggest that, according to eq. (73) in Tarasov2003 [13], one has to set $b_3 = 0$.

Summary

- **We derived a new recursion relation for one-loop scalar Feynman integrals:** self-energies, vertices, boxes etc.
- The condition $\nu_i = 1$ seems to be essential for that.
- A generalization to multiloops seems to be not straightforward or impossible.
- **Solving the recursions for self-energies, vertices** in terms of special functions (and for boxes, not shown here) reproduces essential parts of the results of Tarasov et al. from 2003.
- **Concerning their b_3-terms, we see a need of improvement compared to their paper, if their result is not just wrong in some kinematical situations.** Our conclusions concerning that depend somewhat on an interpretation of their text.
- **We derived a new series of Mellin-Barnes representations: 1-dimensional for self-energies, 2-dim. for vertices, and 3-dimensional for box diagrams** for the most general kinematics. Compared to dim=3, 5, 9 respectively, in the "conventional" Mellin-Barnes-approach.
  This is not yet worked out. Again, we see no direct generalization to multi-loops.
- The special case of **vanishing Gram determinant** $G_n = 0$ is not covered. But small Gram determinants are, and one has to take measures to get reasonable numerics. $\rightarrow$ Small Gram dets are very interesting, but nothing is done.

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