Scalar one-loop Feynman integrals in arbitrary space-time dimension

Tord Riemann, DESY  Work done together with: J. Blümlein and Dr. Phan

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Why one-loop Feynman integrals?  
And why in $D = 4 + 2n - 2\epsilon$ dimensions?

Based on [1, 2], I began in 1980 to calculate Feynman integrals, see Mann, Riemann, 1983 [3]: “Effective Flavor Changing Weak Neutral Current In The Standard Theory And Z Boson Decay”

**Basics**

The seminal papers on 1-loop Feynman integrals:
’t Hooft, Veltman, 1978 [1]: “Scalar oneloped integrals”
Passarino, Veltman, 1978 [2]: “One Loop Corrections for $e^+e^- \textup{ Annihilation into } \mu^+\mu^-$ in the Weinberg Model”

**Interest in “modern” developments for the calculation of 1-loop integrals from basically two sides**

1. For many-particle calculations, there appear inverse Gram determinants from tensor reductions in the answers.

These $1/G(p_i)$ may diverge, because Gram dets can exactly vanish: $G(p_i) \equiv 0$.

One may perform tensor reductions so that no inverse Grams appear, but one has to buy 1-loop integrals in higher dimensions, $D = 4 + 2n - 2\epsilon$. See [4, 5]
Interest in “modern” developments for the calculation of 1-loop integrals from basically two sides

2. Higher-order loop calculations need h.o. contributions from $\epsilon$-expansions of 1-loops:

$$\frac{1}{(d-4)} = -\frac{1}{2\epsilon}$$

A Seminal paper was on $\epsilon$-terms of 1-loop functions:

Nierste, Müller, Böhm, 1992 [6]: “Two loop relevant parts of D-dimensional massive scalar one loop integrals”

1-loop integrals in $D$ dimensions

A general solution in $D$ dimensions was derived in another 2 seminal papers:


I was wondering if the results of Fleischer/Jegerlehner/Tarasov (2003) are sufficiently general for practical, black-box applications, and saw a need of creating a software solution in terms of contemporary mathematics.
So we decided to study the issue from scratch in 2 steps:

1st step: Re-derive analytical expressions for scalar one-loop integrals as meromorphic functions of arbitrary space-time dimension \( D \)

- 2-point functions: Gauss hypergeometric functions \( _2F_1 \) [9]
- 3-point functions: additional Kamp’e de F’eriet functions \( F_1 \); there are the Appell functions \( F_1, \ldots F_4 \) [10]
- 4-point functions: additional Lauricella-Saran functions \( F_5 \) [11]

2nd step:
Derive the Laurent expansions around the singular points of these functions.

This talk:
- Analytical expressions for self-energies, vertices, boxes
- Numerical checks
Feynman integrals

A new recursion for $J_n$

2-point

3-point

Vertex numerics

4-point

Summary

References

Introduction

$J_N \equiv J_N(d; \{p_ip_j\}, \{m^2_i\}) = \int \frac{d^d k}{i \pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_N^{\nu_N}}$  

(1)

with

$D_i = \frac{1}{(k + q_i)^2 - m^2_i + i \epsilon}$.  

(2)

$\nu_i = 1$, $\sum_{i=1}^{n} p_i = 0$  

(3)
\[
J_n = (-1)^n \Gamma(n - d/2) \int_0^1 \prod_{j=1}^n dx_j \delta \left(1 - \sum_{i=1}^n x_i \right) \frac{1}{F_n(x)^{n-d/2}} \tag{4}
\]

Here, the \( F \)-function is the second Symanzik polynomial.

It is derived from the propagators (2),

\[
M^2 = x_1 D_1 + \cdots + x_N D_N = k^2 - 2Qk + J. \tag{5}
\]

Using \( \delta(1 - \sum x_i) \) under the integral in order to transform linear terms in \( x \) into quadratic ones, we may obtain:

\[
F_n(x) = -(\sum_i x_i) J + Q^2 = \frac{1}{2} \sum_{i,j} x_i Y_{ij} x_j - i\epsilon, \tag{6}
\]

The \( Y_{ij} \) are elements of the Cayley matrix, introduced for a systematic study of one-loop \( n \)-point Feynman integrals e.g. in [12]

\[
Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2. \tag{7}
\]

There are \( N_n = \frac{1}{2} n(n + 1) \) different \( Y_{ij} \) for \( n \)-point functions: \( N_3 = 6, N_4 = 10, N_5 = 15 \).
The operator $k^-$ . . .

. . . will reduce an $n$-point Feynman integral $J_n$ to an $(n - 1)$-point integral $J_{n-1}$ by shrinking the propagator $1/D_k$

$$k^- J_n = k^- \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{\prod_{j=1}^n D_j} = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{\prod_{j\neq k,j=1}^n D_j}.$$  \quad (8)

Mellin-Barnes representation

$$\frac{1}{(1+z)^\lambda} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\lambda+s)}{\Gamma(\lambda)} z^s = \sum_{\lambda} \frac{\Gamma(b)}{\Gamma(1-b)} \Gamma(b) \sum_{\lambda} \frac{\Gamma(b)}{\Gamma(1-b)}.$$  \quad (9)

It is valid if $|\text{Arg}(z)| < \pi$ and the integration contour has to be chosen such that the poles of $\Gamma(-s)$ and $\Gamma(\lambda+s)$ are well-separated. The right hand side of (9) is identified as Gauss’ hypergeometric function. For more details see [13]).
F-function and Gram and Cayley determinants

Gram and Cayley det’s are introduced by Melrose [12] (1965). The Cayley determinant $\lambda_{12\ldots N}$ is composed of the

$$Y_{ij} = m_i^2 + m_j^2 - (q_i - q_j)^2$$

introduced in (7), and its determinant is:

$$\lambda_n \equiv \lambda_{12\ldots n} = \begin{vmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{12} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{vmatrix}.$$ (10)

We also define the $(n - 1) \times (n - 1)$ dimensional Gram determinant $g_n \equiv g_{12\ldots n}$,

$$G_n \equiv G_{12\ldots n} = -\begin{vmatrix} (q_1 - q_n)^2 & (q_1 - q_n)(q_2 - q_n) & \cdots & (q_1 - q_n)(q_{n-1} - q_n) \\ (q_1 - q_n)(q_2 - q_n) & (q_2 - q_n)^2 & \cdots & (q_2 - q_n)(q_{n-1} - q_n) \\ \vdots & \vdots & \ddots & \vdots \\ (q_1 - q_n)(q_{n-1} - q_n) & (q_2 - q_n)(q_{n-1} - q_n) & \cdots & (q_{n-1} - q_n)^2 \end{vmatrix}.$$ (11)

Both determinants are independent of a common shifting of the momenta $q_i$. Further, the Gram det $G_n$ is independent of the propagator masses.
One further notation will be introduced, namely that of co-factors of the Cayley matrix. Also called signed minors in e.g. [12, 14]):

\[
\begin{pmatrix}
  j_1 & j_2 & \cdots & j_m \\
  k_1 & k_2 & \cdots & k_m \\
\end{pmatrix}_n.
\]

(12)

The signed minors are determinants, labeled by those rows \( j_1, j_2, \cdots j_m \) and columns \( k_1, k_2, \cdots k_m \) which have been discarded from the definition of the Cayley determinant \((\cdot)_n\), with a sign convention.

\[
\text{sign} \begin{pmatrix}
  j_1 & j_2 & \cdots & j_m \\
  k_1 & k_2 & \cdots & k_m \\
\end{pmatrix}_n = (-1)^{j_1 + j_2 + \cdots + j_m + k_1 + k_2 + \cdots + k_m} \times \text{Signature}[j_1, j_2, \cdots j_m] \times \text{Signature}[k_1, k_2, \cdots k_m].
\]

(13)

Here, \( \text{Signature} \) (defined like the Mathematica command) gives the sign of permutations needed to place the indices in increasing order.

Cayley matrix, by definition:

\[
\lambda_n = \begin{pmatrix}
  0 \\
  0 \\
\end{pmatrix}_n.
\]

(14)

Further, it is (see [15]):

\[
-\frac{1}{2} \partial_i \lambda_n \equiv -\frac{1}{2} \frac{\partial \lambda_n}{\partial m_i^2} = \begin{pmatrix}
  0 \\
  i \\
\end{pmatrix}_n.
\]

(15)
Rewriting the $F$-function further, exploring the $x_n = 1 - \sum x_i$ ...

The elimination of one of the $x_i$ creates linear terms in $F(x)$.

\begin{equation}
F_n(x) = x^T G_n x + 2H_n^T x + K_n.
\end{equation}

The $F_n(x)$ may be cast by shifts $x \to (x - y)$ into the form

\begin{equation}
F_n(x) = (x - y)^T G_n (x - y) + r_n - i\varepsilon = \Lambda_n(x) + r_n - i\varepsilon = \Lambda_n(x) + R_n,
\end{equation}

\begin{equation}
\Lambda_n(x) = (x - y)^T G_n (x - y),
\end{equation}

and

\begin{equation}
\begin{aligned}
r_n &= K_n - H_n^T G_n^{-1} H_n = -\frac{\lambda_n}{g_n} =! - \begin{pmatrix} 0 \\ 0 \end{pmatrix}^n.
\end{aligned}
\end{equation}

The inhomogeneity $R_n = r_n - i\varepsilon$ carries the $i\varepsilon$-prescription.
The linear shifts $y_i$

The $(n-1)$ components $y_i$ of the vector $y$ appearing here in $F_n(x)$ are:

$$y_i = - \left( G^{-1}_n K_n \right)_i, \quad i \neq n \quad (20)$$

The following relations are also valid:

$$y_i = \frac{\partial r_n}{\partial m_i^2} = - \frac{1}{g_n} \frac{\partial \lambda_n}{\partial m_i^2} = - \frac{\partial_i \lambda_n}{g_n} = \frac{2}{g_n} \left( \begin{array}{c} 0 \\ i \end{array} \right)_n, \quad i = 1 \cdots n. \quad (21)$$

The auxiliary condition $\sum_i^n y_i = 1$ is fulfilled.

We see that the notations for the $F$-function are finally independent of the choice of the variable which was eliminated by use of the $\delta$-function in the integrand of (4). The inhomogeneity $R_n$ is the only variable carrying the causal $i\epsilon$-prescription, while e.g. $\Lambda(x)$ and the $y_i$ are by definition real quantities.
The recursion relation for $J_n$

One may use the Mellin-Barnes relation (9) in order to decompose the integrand of $J_n$ given in (4) as follows:

$$\frac{1}{[F(x)]^{n-d/2}} \equiv \frac{1}{[\Lambda_n(x) + R_n]^{n-d/2}} \equiv \frac{R_n^{-(n-d/2)}}{[1 + \frac{\Lambda_n(x)}{R_n}]^{n-d/2}}$$

$$= \frac{R_n^{-(n-d/2)}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(n - \frac{d}{2} + s)}{\Gamma(n - \frac{d}{2})} \left[ \frac{\Lambda_n(x)}{R_n} \right]^s,$$

(22)

for $|\text{Arg}(\Lambda_n/R_n)| < \pi$. The condition always applies. Further, the integration path in the complex $s$-plane separates the poles of $\Gamma(-s)$ and $\Gamma(n - \frac{d}{2} + s)$. As a result of (22), the Feynman parameter integral of $J_n$ becomes homogeneous:

$$K_n = \prod_{j=1}^{n-1} \int_0^{1-\sum_{i=j+1}^{n-1} x_i} dx_j \left[ \frac{\Lambda_n(x)}{R_n} \right]^s \equiv \int dS_{n-1} \left[ \frac{\Lambda_n(x)}{R_n} \right]^s.$$

(23)
The recursion relation for $J_n$

In order to solve the integral in (23), we consider the differential operator $\hat{P}_n$ [16, 17],

$$\hat{P}_n \left[ \frac{\Lambda_n(x)}{R_n} \right]^s \equiv \sum_{i=1}^{n-1} \frac{1}{2} (x_i - y_i) \frac{\partial}{\partial x_i} \left[ \frac{\Lambda_n(x)}{R_n} \right]^s = s \left[ \frac{\Lambda_n(x)}{R_n} \right]^s. \quad (24)$$

This eigenvalue relation allows to introduce the operator $\hat{P}_n$ into the integrand of (23):

$$K_n = \int dS_{n-1} \frac{\hat{P}_n}{s} \left[ \frac{\Lambda_n(x)}{R_n} \right]^s = \frac{1}{2s} \sum_{i=1}^{n-1} \prod_{k=1}^{n-1} \int_0^{u_k} dx' (x_i - y_i) \frac{\partial}{\partial x_i} \left[ \frac{\Lambda_n(x)}{R_n} \right]^s. \quad (25)$$

After a series of manipulations in order to perform one of the $x$-integrations – by partial integration, eating the corresponding differential – one arrives at:

$$J_n = \left( -1 \right)^n \frac{n}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(n - \frac{d}{2} + s) \Gamma(s + 1)}{2 \Gamma(s + 2)} \left( \frac{1}{R_n} \right)^{n - \frac{d}{2}} \times \sum_{i=1}^{n} \left\{ \frac{\partial r_n}{\partial m_i^2} \right\} \int dS_{n-2} \left[ \frac{F_{n-1}^{(i)}}{R_n} - 1 \right]^s \quad (26)$$
We stress again that only the $R_n$ carries an $i\epsilon$. Now it is important to eliminate the term $(-1)$ from the combination $(F_{n-1}^{(i)}/R_n - 1)^s$ under the Mellin-Barnes integral over $s$, because then we arrive at a sum over the $n$ different $(n-1)$-point functions arising from skipping a propagator from the original integral. In fact, this may be arranged using the following relation for $(-z) = F/R - 1$ [18]:

$$
\frac{1}{\Gamma(c + s)} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a + s) \Gamma(b + s) \Gamma(c + s)}{\Gamma(-s) \Gamma(a + b - c - s) \Gamma(c - a + s) \Gamma(c - b + s) \Gamma(c - a) \Gamma(c - b)} (-z)^s
$$

(27)

provided that $|\text{Arg}(-z)| < 2\pi$.

We arrive at the following recursion relation:
The recursion relation for 1-loop $n$-point functions

\[ J_n(d, \{q_i, m_i^2\}) = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\frac{d-n+1}{2} + s)\Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} R_{n}^{-s} \times \sum_{k=1}^{n} \left( \frac{1}{r_n} \frac{\partial r_n}{\partial m_k^2} \right) \Delta^k J_n(d + 2s; \{q_i, m_i^2\}). \] \hspace{1cm} (28)

The cases $G_n = 0$ and $\lambda_n = r_n = 0$ prevent the use of the Mellin-Barnes transformation. They are simpler than what we have to do here. Details are given elsewhere.

1-point function, or tadpole

\[ J_1(d; m^2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 - m^2 + i\epsilon} = -\frac{\Gamma(1 - d/2)}{(m^2 - i\epsilon)^{1-d/2}}. \] \hspace{1cm} (29)

Comments

1. In Tarasov 2003 [8], a recursion was derived where our Mellin-Barnes integral is replaced by an infinite sum to be solved. Formulae for 2,3,4-point functions are given.
2. A 4-point function is a 3-fold integral. With AMBRE, we get up to 15-fold integrals instead.
3. See Johann Usovitsch’s talk: Integrand is equally integrable for Euklidean and Minkoswkian cases. No Gram=0 problem.
The 2-point function

From our recursion relation (28), taken at \( n = 2 \) and using the expression (29) with \( d \rightarrow d + 2s \) for the one-point functions under the integral, one gets the following representation:

\[
J_2(D; q_1, m_1^2, q_2, m_2^2) = e^{\epsilon \gamma_E} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma\left(\frac{D-1}{2} + s\right) \Gamma(s+1)}{2 \Gamma\left(\frac{D-1}{2}\right)} R_s^2
\times \left[ \frac{1}{r_2} \frac{\partial r_2}{\partial m_2^2} \left( \frac{1 - \frac{D+2s}{2}}{(m_1^2)^{1-\frac{D+2s}{2}}} + (m_1^2 \leftrightarrow m_2^2) \right) \right].
\] (30)

One may close the integration contour of the MB-integral in (30) to the right, apply the Cauchy theorem and collect the residua originating from two series of zeros of arguments of \( \Gamma \)-functions at \( s = m \) and \( s = m - d/2 - 1 \) for \( m \in \mathbb{N} \).

The first series stems from the MB-integration kernel, the other one from the dimensionally shifted 1-point functions.

And then summing up in terms of Gauss’ hypergeometric functions.
The 2-point function (slightly rewritten), \( R_2 \equiv R_{12} \)

\[
J_2(d; Q^2, m_1^2, q_2, m_2^2) = -\frac{\Gamma\left(2 - \frac{d}{2}\right) \Gamma\left(d - 1\right)}{(d - 2) \Gamma\left(\frac{d}{2}\right) R_2} \frac{\partial_2 R_2}{R_2} \\
\left[ (m_1^2)^{\frac{d}{2} - 1} \right. \\
\left. \quad \binom{1}{\frac{d}{2} - \frac{1}{2}; \frac{m_1^2}{R_2}} + \frac{R_2^{\frac{d}{2} - 1}}{\sqrt{1 - \frac{m_1^2}{R_2}}} \sqrt{\pi} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(d - \frac{1}{2}\right)} \right]
\]

\[ + (m_1^2 \leftrightarrow m_2^2) \]

The representation (31) is valid for \( \left| \frac{m_1^2}{r_{12}} \right| < 1, \left| \frac{m_2^2}{r_{12}} \right| < 1 \) and \( \Re\left(\frac{d-2}{2}\right) > 0 \).

The result is in agreement with Eqn. (53) of Tarasov et al. (2003) [8].
The 3-point function

According to the master formula (28), we can write the massive 3-point function as a sum of three terms:

\[ J_3 = J_{123} + J_{231} + J_{312}, \]  

(32)

using the representation for e.g. \( J_{123} \)

\[
J_{123}(d, \{q_i, m_i^2\}) = -\frac{e^{\epsilon \gamma E}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma\left(\frac{d-2+2s}{2}\right) \Gamma(s + 1)}{2 \Gamma\left(\frac{d-2}{2}\right)} R_3^{-s} \\
\times \frac{1}{r_3} \frac{\partial r_3}{\partial m_3^2} J_2(d + 2s; q_1, m_1^2, q_2, m_2^2). \]

(33)
Here, $J_2(d + 2s; q_1, m_1^2, q_2, m_2^2)$ is given by (31), taken at $d + 2s$ dimensions. By performing the Mellin-Barnes integrals, one gets three terms, each consisting of eight series, from taking the residues by closing the integration contours to the right; one of the three terms is:

$$J_{123} = \Gamma \left( 2 - \frac{d}{2} \right) R_{123}^{\frac{d}{2} - 2} \times b_{123}$$

$$- \sqrt{\pi} \Gamma \left( 2 - \frac{d}{2} \right) \frac{\Gamma \left( \frac{d}{2} - 1 \right)}{\Gamma \left( \frac{d-1}{2} \right)} \frac{\partial_3 \lambda_{123}}{\lambda_{123}} \frac{R_{12}^{d-1} \left( \frac{d}{2} - 1 \right)}{4 \lambda_{12}} \left[ \frac{\partial_2 \lambda_{12}}{\sqrt{1 - \frac{m_1^2}{R_{12}^2}}} + \frac{\partial_1 \lambda_{12}}{\sqrt{1 - \frac{m_2^2}{R_{12}^2}}} \right] \times 2F1 \left[ \frac{d-2}{d-1}, 1; \frac{R_{12}}{R_{123}} \right]$$

$$+ \frac{2}{d-2} \Gamma \left( 2 - \frac{d}{2} \right) \frac{\partial_3 \lambda_{123}}{\lambda_{123}} \times \left[ \frac{\partial_2 \lambda_{12}}{\sqrt{1 - \frac{m_1^2}{R_{12}^2}}} \right] F1 \left( \frac{d-2}{2}; 1, \frac{1}{2}, 2 \right) \frac{m_1^2}{R_{123}} \frac{m_2^2}{R_{12}}$$

$$+ (m_1^2 \leftrightarrow m_2^2),$$

and

$$b_{123} = - \frac{1}{2 g_{12}} \frac{\partial_3 \lambda_{123}}{\lambda_{123}} \left( \frac{\partial_2 \lambda_{12}}{\sqrt{1 - \frac{m_1^2}{R_{12}^2}}} + \frac{\partial_1 \lambda_{12}}{\sqrt{1 - \frac{m_2^2}{R_{12}^2}}} \right) 2F1 \left[ \frac{1}{2}, \frac{1}{2}; \frac{R_{12}}{R_{123}} \right]$$

$$- \frac{\partial_3 \lambda_{123}}{\lambda_{123}} \left\{ \frac{\partial_2 \lambda_{12}}{\sqrt{1 - \frac{m_1^2}{R_{12}^2}}} \frac{m_1^2}{4 \lambda_{12}} F1 \left( 1; \frac{1}{2}, 2; \frac{m_1^2}{R_{123}}, \frac{m_1^2}{R_{12}} \right) + (m_1^2 \leftrightarrow m_2^2) \right\},$$

$$\text{1-loop-functions in d dimensions}$$
where $\partial_i \lambda_j ...$ is defined in (21). The representation (32) is valid for $\text{Re}(d - 2/2) > 0$. The conditions $|m_i^2/R_{ij}| < 1$, $|R_{ij}/R_{ijk}| < 1$ had to be met during the derivation. The result may be analytically continued in a straightforward way, however, in the complete complex domain. The functions $\binom{2}{1}$ and $\binom{1}{1}$ of the $b_{ijk}$-terms are met by setting $d = 4$ in the corresponding functions $J_{ijk}$ of the general $J_3$. 


An alternative writing of $J_3 = J_{123} + J_{231} + J_{312}$ is, with $R_3 = R_{123}, R_2 = R_{12}$ etc. here:

The massive vertex

$$J_{123} = \Gamma \left( 2 - \frac{d}{2} \right) \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{r_2}{2 \sqrt{1 - m_1^2/r_2}}$$

$$- R_2^{d/2 - 2} \frac{\sqrt{\pi}}{2} \frac{\Gamma \left( \frac{d}{2} - 1 \right)}{\Gamma \left( \frac{d}{2} - \frac{1}{2} \right)} 2F_1 \left[ \frac{d-2}{2} - \frac{1}{2} ; \frac{R_2}{R_3} \right] + R_3^{d/2 - 2} 2F_1 \left[ \frac{1}{3} - \frac{1}{2} ; \frac{R_2}{R_3} \right]$$

$$+ \Gamma \left( 2 - \frac{d}{2} \right) \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{m_1^2}{4 \sqrt{1 - m_1^2/r_2}}$$

$$+ \frac{2(m_1^2)^{d/2 - 2}}{d - 2} F_1 \left( \frac{d - 2}{2} ; 1, \frac{1}{2} ; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2} \right) - R_3^{d/2 - 2} F_1 \left( 1, 1, \frac{1}{2} ; \frac{m_1^2}{R_3}, \frac{m_2^2}{R_2} \right)$$

$$+ (m_1^2 \leftrightarrow m_2^2)$$

For $d \to 4$, both the [...] approach zero.

So the $J_3$ is finite in this limit, as it should be for massive 3-point function.
For the 3-point function, we look at the expression \( J_{123} + J_{231} + J_{312} \).

We should agree with Eqn. (74) to (76) of Tarasov (2003).

Our terms with \( d \)-dimensional \( F_1 \) and \( _2F_1 \) do agree exactly, but \( b_{123} + b_{231} + b_{312} \) looks quite different.

**Tarasov (2003) [8], Eqns. (73) and (75)**

There are kinematic conditions on internal momenta \( q_{ij}^2 = (q_i - q_j)^2 \) to be respected; the \( b_3 \)-term of Tarasov becomes:

\[
J_3(b_3) = \theta(-G_3) \times \theta(q_{ij}^2) \times \theta\left(\frac{m_i^2}{r_3} - 1\right) \\
\times \frac{\Gamma(2-d/2)}{\lambda_3} \left(2^{3/2} \pi \sqrt{-G_3} \ R_3^{d/2-1}\right)
\]

(36)

**Otherwise:**

\[
J_3(b_3) = b_3 = 0.
\]

(37)
### Numerics for 3-point functions, table 1

<table>
<thead>
<tr>
<th>([p_i^2], [m_i^2])</th>
<th>([+100, +200, +300], [10, 20, 30])</th>
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<td>(\lambda_{123})</td>
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</tr>
<tr>
<td>(\sum J)-terms</td>
<td>((0.019223879 - 0.007987267 I))</td>
</tr>
<tr>
<td>(\sum b_3)-terms</td>
<td>0 ((0.019223879 - 0.007987267 I))</td>
</tr>
<tr>
<td>(J_3) (TR)</td>
<td>(0.0192238790286244077-0.00798726725497102795) i</td>
</tr>
</tbody>
</table>

\[b_3\]-term: \((-0.089171509 + 0.069788641 I)\) \(+ (0.022214414 I)/\epsilon\)  
\[b_3 + \sum J\]-terms: \((-0.0123077377 - 0.009301346 I)\)  
\(J_3\) (OT) \(\sum J\)-terms, \(b_3\)-term \(\rightarrow 0\), OK

**MB suite**

\((-1)^*\text{fiesta3}\) \(-0.012307 + 0.009301 I\) \(+ (8*10^{-6} + 0.00001 I) \text{ pm4} \)  

**LoopTools/FF, \(\epsilon^0\)** \(0.0192238790286244077-0.00798726725497102795\) i

**Table 1**: Numerics for a vertex in space-time dimension \(d = 4 - 2\epsilon\). Causal \(\epsilon = 10^{-20}\). Red input quantities (external momenta shown here!) suggest that, according to Eqn. (73) in Tarasov (2003) [8], one has to set \(b_3 = 0\).

Although \(b_3\) of [8] deviates from our vanishing value, it has to be set to zero, \(b_3 \rightarrow 0\).

**The results of both calculations for \(J_3\) agree for this case.**
### Numerics for 3-point functions, table 2

<table>
<thead>
<tr>
<th>([p_i^2], [m_i^2])</th>
<th>([-100, +200, -300]), ([10, 20, 30])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_{123})</td>
<td>480000</td>
</tr>
<tr>
<td>(\lambda_3)</td>
<td>-19300000</td>
</tr>
<tr>
<td>(m_i^2/r_3)</td>
<td>0.248705, 0.497409, 0.746114</td>
</tr>
<tr>
<td>(m_i^2/r_{12})</td>
<td>0.248447, 0.496894, 0.745342</td>
</tr>
<tr>
<td>(m_i^2/r_{23})</td>
<td>-0.39801, -0.79602, -1.19403</td>
</tr>
<tr>
<td>(m_i^2/r_{31})</td>
<td>0.104895, 0.20979, 0.314685</td>
</tr>
<tr>
<td>(\sum J\text{-terms})</td>
<td>(-0.012307377 - 0.056679689 I) + ( + 0.012825498 I)/(\epsilon)</td>
</tr>
<tr>
<td>(\sum b_3\text{-terms})</td>
<td>( + 0.047378343 I) - ( + 0.012825498 I)/(\epsilon)</td>
</tr>
<tr>
<td>(J_3\text{(TR)})</td>
<td>(-0.012307377 - 0.009301346 I) - ( + 0.012825498 I)/(\epsilon)</td>
</tr>
<tr>
<td>(b_3)-term</td>
<td>( + 0.047378343 I) - ( + 0.012825498 I)/(\epsilon)</td>
</tr>
<tr>
<td>(b_3 + \sum J\text{-terms})</td>
<td>(-0.012307377 - 0.009301346 I) - ( + 0.012825498 I)/(\epsilon)</td>
</tr>
<tr>
<td>(J_3\text{(OT)})</td>
<td>(\sum J\text{-terms, }b_3\text{-term} \rightarrow 0, \text{ gets wrong})</td>
</tr>
</tbody>
</table>

#### MB suite

\((-1)\text{fiesta3}\) & \((-0.012307 + 0.009301 I) + (8*10^{-6} + 0.00001 I) pm4\) & & |

\(\text{LoopTools/FF, }\epsilon^0\) & \(-0.0123073773677820630 - 0.0093013461700863289 I\)

**Table 2:** Numerics for a vertex in space-time dimension \(d = 4 - 2\epsilon\). Causal \(\epsilon = 10^{-20}\). Red input quantities suggest that, according to eq. (73) in Tarasov2003 [8], one has to set \(b_3 = 0\). Further, we have set in the numerics for eq. (75) of Tarasov2003 [8] that \(\text{Sqrt}\{-g_{123} + I*\text{epsilon}\}\), what looks counter-intuitive for a “momentum”-like function.

**Both results agree if we do not set Tarasov’s \(b_3 \rightarrow 0\).**
### Numerics for 3-point functions, table 3

<table>
<thead>
<tr>
<th>( p_i^2 )</th>
<th>-100, -200, -300</th>
<th>( m_i^2 )</th>
<th>10, 20, 30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_{123} )</td>
<td>-160000</td>
<td>( \lambda_{123} )</td>
<td>15260000</td>
</tr>
<tr>
<td>( m_i^2 / r_{123} )</td>
<td>0.104849, 0.209699, 0.314548</td>
<td>( m_i^2 / r_{12} )</td>
<td>0.248447, 0.496894, 0.745342</td>
</tr>
<tr>
<td>( m_i^2 / r_{23} )</td>
<td>0.133111, 0.266223, 0.399334</td>
<td>( m_i^2 / r_{31} )</td>
<td>0.104895, 0.20979, 0.314685</td>
</tr>
<tr>
<td>( \sum J )-terms</td>
<td>(0.0933877 – 0 i)</td>
<td>( \sum b )-terms</td>
<td>-0.101249</td>
</tr>
<tr>
<td>( J_3 ) (TR)</td>
<td>(-0.00786155 – 0 i)</td>
<td>( b_3 )</td>
<td>(-0.101249 + 0 i)</td>
</tr>
<tr>
<td>( b_3 + J )-terms</td>
<td>(-0.007861546 + 0 i)</td>
<td>( J_3 ) (OT)</td>
<td>( b_3 + J )-terms ( \rightarrow ) OK</td>
</tr>
<tr>
<td>MB suite</td>
<td>-0.007862014, 5.002549159*10-6, 0</td>
<td>(-1)*fiesta3</td>
<td>-(0.007862)</td>
</tr>
<tr>
<td>( \text{LoopTools/FF, } \epsilon^0 )</td>
<td>-0.00786154613229082290</td>
<td></td>
<td>+ (6<em>10-6 + 6</em>10-6 \pm 10)</td>
</tr>
</tbody>
</table>

Table 3: Numerics for a vertex in space-time dimension \( d = 4 – 2\epsilon \). Causal \( \epsilon = 10^{-20} \). Agreement with Tarasov (2003).
### Numerics for 3-point functions, table 4

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_i^2 )</td>
<td>+100, −200, +300</td>
</tr>
<tr>
<td>( m_i^2 )</td>
<td>10, 20, 30</td>
</tr>
<tr>
<td>( G_{123} )</td>
<td>4800000</td>
</tr>
<tr>
<td>( \lambda_{123} )</td>
<td>4900000</td>
</tr>
<tr>
<td>( m_i^2/r_{123} )</td>
<td>−0.979592, −1.95918, −2.93878</td>
</tr>
<tr>
<td>( m_i^2/r_{12} )</td>
<td>−0.97561, −1.95122, −2.92683</td>
</tr>
<tr>
<td>( m_i^2/r_{23} )</td>
<td>0.133111, 0.266223, 0.399334</td>
</tr>
<tr>
<td>( m_i^2/r_{31} )</td>
<td>−0.180723, −0.361446, −0.542169</td>
</tr>
<tr>
<td>( \sum J)-terms</td>
<td>(0.006243624 - 0.018272524 I)</td>
</tr>
<tr>
<td>( \sum b_3)-terms</td>
<td>0</td>
</tr>
<tr>
<td>( J_3(\text{TR}) )</td>
<td>(0.006243624 - 0.018272524 I)</td>
</tr>
<tr>
<td>( b_3)-term</td>
<td>(0.040292491 + 0.029796253 I)</td>
</tr>
<tr>
<td>( b_3 + \sum J)-terms</td>
<td>(-0.012307377 - 0.009301346 I)</td>
</tr>
<tr>
<td>( J_3(\text{OT}) )</td>
<td>( \sum J)-terms, ( b_3)-term ( \to ) 0, OK</td>
</tr>
<tr>
<td>MB suite</td>
<td></td>
</tr>
<tr>
<td>(-1)*fiesta3</td>
<td>-(-0.006322 + 0.014701 I)</td>
</tr>
<tr>
<td>LoopTools/FF, ( \epsilon^0 )</td>
<td>0.00624362477277410 - 0.01827252404872805 i</td>
</tr>
</tbody>
</table>

**Table 4:** Numerics for a vertex in space-time dimension \( d = 4 - 2\epsilon \). Causal \( \epsilon = 10^{-20} \). Red input quantities suggest that, according to eq. (73) in Tarasov2003 [8], one has to set \( b_3 = 0 \).

**Agreement with Tarasov (2003) due to setting** \( b_3 = 0 \) **there.**
The 4-point function

According to the master formula (28), we can write the massive 4-point function as a sum of four terms:

\[ J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}, \]  

(38)

Each of the four terms has the structure

\[ J_{1234} = \frac{\Gamma \left(2 - \frac{d}{2}\right) \Gamma \left(\frac{d}{2} - 1\right)}{\Gamma \left(\frac{d-3}{2}\right)} \times (r_{1234})^{\frac{d-2}{2}} \times \hat{b}_{1234} \]

\[ + \Gamma (2 - d/2) \times \hat{J}_{1234}^d \]  

(39)

The pre-factor is singular: \( \Gamma(2 - d/2) = 1/\epsilon + \cdots \) for \( d \geq 4 - 2\epsilon \).

We agree for \( \hat{J}_{1234}^d \) etc. with Tarasov (2003) [?].

For the \( b_4 \)-term, the situation is a bit unclear.
The boundary term \( \hat{b}_{1234} \) is independent of \( d \):

\[
\hat{b}_{1234} = \frac{1}{2} \left( \frac{b_{123}}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) \frac{\sqrt{\pi}}{\sqrt{1 - r_{123}/r_{1234}}} \\
+ \sqrt{\pi} \left( \frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) \left( \frac{1}{r_{123}} \frac{\partial r_{123}}{\partial m_3^2} \right) \left( \frac{1}{4 g_{12}} \right) \times \\
\times \left[ \frac{\partial_2 \lambda_{12}}{\sqrt{1 - m_1^2/r_{12}}} + \frac{\partial_1 \lambda_{12}}{\sqrt{1 - m_2^2/r_{12}}} \right] \left( \frac{1}{\sqrt{1 - r_{12}/r_{123}}} \right) F_1 \left( \frac{1}{2}; 1, \frac{1}{2}; 3; \frac{r_{12}}{r_{1234}}, \frac{r_{12}}{r_{123}} \right) \tag{40}
\]

\[
+ \sqrt{\pi} \left( \frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) \left( \frac{1}{r_{123}} \frac{\partial r_{123}}{\partial m_3^2} \right) \times \\
\times \left[ \left( \frac{\partial_2 \lambda_{12}}{1 - m_1^2/r_{12}} \right) \left( \frac{m_1^2}{8 \lambda_{12}} \right) \left( \frac{r_{123}}{r_{123} - m_1^2} \right) \\
\times F_S \left( \frac{1}{2}, 1, 1; 1, \frac{1}{2}; 2, 2, 2; \frac{m_1^2}{r_{1234}}, \frac{m_1^2}{m_1^2 - r_{123}}, \frac{m_1^2}{m_1^2 - r_{12}} \right) \right] + (1 \leftrightarrow 2) \\
+(2, 3, 1) + (3, 1, 2).
\]

The boundary term \( b_4 \) has not been exactly defined in [?], concerning the kinematical conditions. We did not perform massive numerical tests.
and

\[ \hat{J}_{1234} = (r_{1234})^{\frac{d}{2} - 2} \times b_{1234} \]

\[ - \frac{1}{2} \left( \frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m^2_4} \right) \binom{d - 3}{d - 2} + \Gamma \left( \frac{d}{2} - 1 \right) \left( \binom{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m^2_4} \right) \left( \binom{1}{r_{123}} \frac{\partial r_{123}}{\partial m^2_3} \right) \left[ \frac{\partial_2 \lambda_{12}}{\sqrt{1 - \frac{m^2_1}{r_{12}}} + \sqrt{1 - \frac{m^2_2}{r_{12}}}} \right] \times \]

\[ + \sqrt{\pi} \frac{\Gamma \left( \frac{d}{2} - 1 \right)}{\Gamma \left( \frac{d}{2} \right)} \left( \binom{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m^2_4} \right) \left( \binom{1}{r_{123}} \frac{\partial r_{123}}{\partial m^2_3} \right) \left[ \frac{\partial_2 \lambda_{12}}{\sqrt{1 - \frac{m^2_1}{r_{12}}} + \sqrt{1 - \frac{m^2_2}{r_{12}}}} \right] \times \]

\[ \times \left( \frac{r_{12}^{\frac{d}{2} - 1}}{8 \lambda_{12}} \right) \left( \frac{1}{\sqrt{1 - \frac{r_{12}}{r_{123}}}^{d - 1}} \right) \binom{d}{d - 3} \binom{1}{1, \frac{1}{2}} \binom{d - 1}{r_{12}, r_{123}} \]

\[ - \frac{\Gamma \left( \frac{d}{2} - 1 \right)}{\Gamma \left( \frac{d}{2} \right)} \left( \binom{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m^2_4} \right) \left( \binom{1}{r_{123}} \frac{\partial r_{123}}{\partial m^2_3} \right) \times \]

\[ \times \left[ \left( \frac{r_{123}}{r_{123} - m^2_1} \right) \left( \frac{r_{12}}{r_{12} - m^2_1} \right) \left( \frac{\partial_2 \lambda_{12}}{8 \lambda_{12}} \right) \left( m^2_1 \right)^{\frac{d}{2} - 1} \times \]

\[ \times F_S \left( \frac{d - 3}{2}, 1, 1; \frac{d}{2}, \frac{d}{2}, \frac{d}{2}; \frac{m^2_1}{r_{1234}}, \frac{m^2_1}{r_{123} - r_{12}}, \frac{m^2_1}{r_{123} - r_{12}} \right) + (1 \leftrightarrow 2) \right] \]

\[ + (2, 3, 1) + (3, 1, 2). \]
An alternative writing of $J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}$ is, with $R_4 = R_{1234}, R_3 = R_{123}, R_2 = R_{12}$ etc. here:

**The massive box function**

$$J_{1234} = \Gamma \left( 2 - \frac{d}{2} \right) \frac{\partial_4 r_4}{r_4} \left\{ \right.$$  

$$\left[ \frac{b_{123}}{2} \left( - R_3^{d/2-2} \right)_2 F_1 \left[ \frac{d-3}{2}, 1; \frac{R_2}{R_3} \right] + R_4^{d/2-2} \sqrt{\pi} \frac{\Gamma \left( \frac{d}{2} - 1 \right)}{\Gamma \left( \frac{d}{2} - \frac{3}{2} \right)} \frac{1}{2} F_1 (d \to 4) \right]\right.$$  

$$+ \frac{\Gamma \left( \frac{d}{2} - 1 \right)}{\Gamma \left( \frac{d}{2} - \frac{3}{2} \right)} \frac{\Gamma \left( \frac{d}{2} - 1 \right)}{4 \sqrt{1 - m_1^2/R_2}} \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \sqrt{1 - m_1^2/R_2} \left[ \right.$$  

$$+ \frac{R_2^{d/2-2}}{d-3} F_1 \left( \frac{d-3}{2}, 1; \frac{1}{2}, \frac{d-1}{2}; \frac{R_2}{R_4}, \frac{R_2}{R_3} \right) - R_4^{d/2-2} F_1 (d \to 4) \right]\right.$$  

$$m_1^2 \frac{\Gamma \left( \frac{d}{2} - 1 \right)}{8 \Gamma \left( \frac{d}{2} - \frac{3}{2} \right)} \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{r_3}{r_3 - m_1^2} \frac{r_2}{r_2 - m_1^2} \left[ \right.$$  

$$- \left( m_1^2 \right)^{d/2-2} \frac{\Gamma \left( \frac{d}{2} - 3/2 \right)}{\Gamma \left( \frac{d}{2} \right)} F_S(d/2 - 3/2, 1, 1, 1, 1, d/2, d/2, d/2, m_1^2/R_4, m_1^2/R_3, m_1^2/R_2) \right.$$  

$$+ R_4^{d/2-2} \sqrt{\pi} F_S (d \to 4) \right\} + \left( m_1^2 \leftrightarrow m_2^2 \right)$$

For $d \to 4$, all three [...] approach zero. So that the massive $J_4$ gets finite then: OK.
Summary

- We derived a new recursion relation for one-loop scalar Feynman integrals: self-energies, vertices, boxes etc.
- The condition $\nu_i = 1$ seems to be essential for that.
- A generalization to multiloops seems to be not straightforward or impossible.
- Solving the recursions in terms of special functions reproduces essential parts of the results of Tarasov et al. from 2003.
- Concerning their $b_3$-terms, we see a need of improvement compared to their paper, if their result is not just wrong in some kinematical situations. Our conclusions concerning that depend somewhat on an interpretation of their text.
- We derived a new series of Mellin-Barnes representations: 1-dimensional for self-energies, 2-dim. for vertices, and 3-dimensional for box diagrams for the most general kinematics. Compared to dim=$3, 5, 9$ respectively, in the “conventional” Mellin-Barnes-approach. This is worked out by Johann Usovitsch. Again, we see no direct generalization to multi-loops.
- The special case of vanishing Gram determinant $G_n = 0$ is not covered. But small Gram determinants are, and one has to take measures to get reasonable numerics. → Small Gram dets are very interesting, but nothing is done.
References


[10] Lauricella functions are generalizations of hypergeometric functions with more than one argument, see http://mathworld.wolfram.com/AppellHypergeometricFunction.html. Among them are $F_{A}^{n}, F_{B}^{n}, F_{C}^{n}, F_{D}^{n}$, studied by Lauricella, and later also by Campe de Ferrie. For $n=2$, these functions become the Appell functions $F_{2}, F_{3}, F_{4}, F_{1}$, respectively, and are the first four in the set of Horn functions. The $F_{1}$ function is implemented in the Wolfram Language as AppellF1[a, b1, b2, c, x, y].


