Feynman diagrams and Mellin-Barnes integrals

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- **Introduction + Motivation**
- **Mathematical Reminder on the Γ-function, residues, the Cauchy-theorem**
- **Derivation of the Feynman integral representation for arbitrary Feynman diagrams**
- **Few simple Feynman integrals, made conventionally**
- **Derivation of Mellin-Barnes representations for Feynman diagrams**
- **How to evaluate them?**
- **The simplest non-trivial case: the massive QED one-loop vertex**
- **Expansions in a small parameter, e.g.** $m^2/s << 1$
- **Non-planar diagrams: Use of Cheng-Wu variables**
- **A numerical approach with the MB suite: AMBRE/MB/MBtools/MBnumerics/CUBA**

Introductory remarks

For many problems of the past, a relatively simple approach to the evaluation of Feynman integrals was sufficient:

★ Tensor reduction a la Passarino/Veltmann [1]

★ Evaluate Feynman parameter integrals by direct integration [2]

Typically 1-loop (massless: 2-loop), typically 2 $\longrightarrow$ 2 scattering (plus bremsstrahlung)

Feynman parameters may be used and by direct integration over them one gets objects like:

$\frac{23}{57}, \ \zeta(3), \ \ln(\frac{t}{s}), \ \ln(\frac{t}{s}) \cdot \ln(\frac{s}{m^2}), \ \mathrm{Li}_2(\frac{t}{s+i\epsilon})$ etc.

With more complexity of the reaction (more legs) and more perturbative accuracy (more loops), this approach appears to be not sufficiently sophisticated.

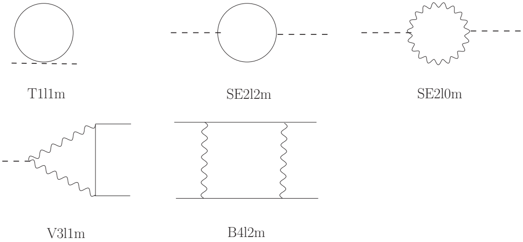


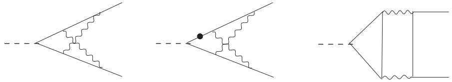
Figure shows so-called master integrals.

$$\texttt{T1l1m} = \frac{1}{\epsilon} + 1 + (1 + \frac{\zeta_2}{2})\epsilon + (1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3})\epsilon^2 +$$

$$\texttt{B4l2m} = [-\frac{1}{\epsilon} + \ln(-s)]\frac{2y\ln(y)}{s(1-y^2)} + c_1\epsilon + \cdots$$

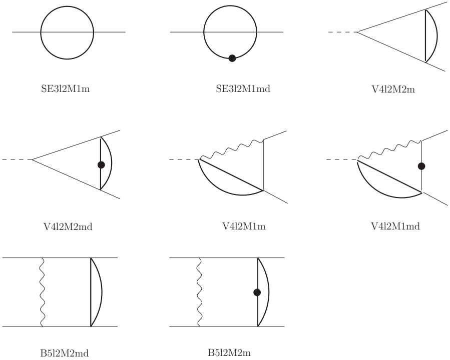
with $d = 4 - 2\epsilon$ and $m = 1$ and $y = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1}$

More loops



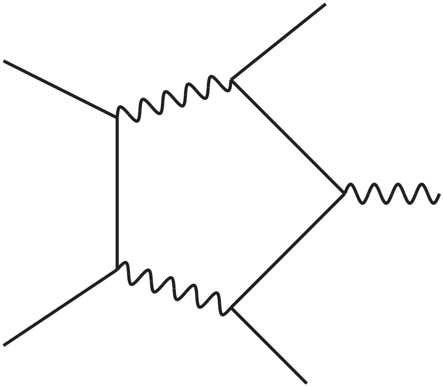
V6l4m1 V6l4m1d V6l4m2

Two-loop vertex integrals with six internal lines
massless case: only fixed numbers and one scale factor

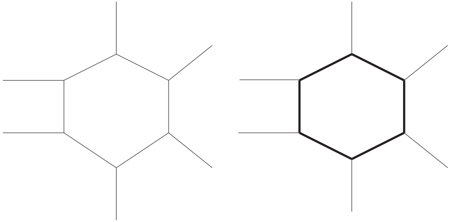


Integrals with two different mass scales m and M

More legs



Massive pentagon: 5 kinematic variables + several masses



Massless and massive hexagons: 8 kinematic variables + several masses

Variables for $2 \to 2$ **scattering, i.e. box diagrams:** s, t **or** s **and** $\cos\theta$

Variables for $2 \to 3$ **scattering: 5 = 2 + 3 (three additional momenta of a particle)**

Variables for $2 \to 4$ **scattering: 8 = 5 + 3 (another three additional)**

Some Mathematical Preparations

We will often use, for $d = 4 - 2\epsilon$:

$$a^{\epsilon} \;=\; e^{\epsilon \ln(a)} = 1 + \ln(a)\,\epsilon + \frac{1}{2}\ln^2(a)\,\epsilon^2 + \ldots \tag{1}$$

The Γ-function

There is no differential equation defining the Γ-function, instead it may be defined by a difference equation:

$$z\Gamma(z) - \Gamma(z+1) = 0 \qquad (2)$$

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty t^{z-1} e^{-t} dt && (3)\\
\Gamma(0) &= \infty && (4)\\
\Gamma(1) &= 1 && (5)\\
\Gamma(n) &= (n-1)!, \quad n = 2, 3, \cdots && (6)
\end{aligned}$$

You might remember that $\Gamma(z)$ has poles at $z = -n, n = 0, 1, 2, 3, \cdots$, and it is

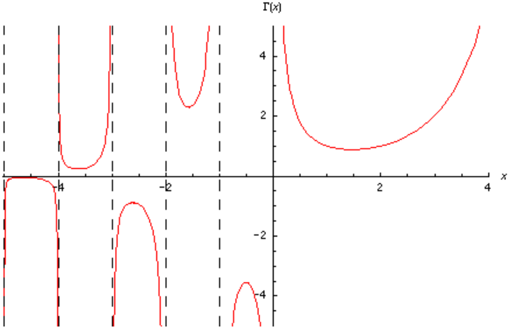
$$\Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_E + \frac{1}{2}\left[{\gamma_E}^2 + \zeta(2)\right]\epsilon + \frac{1}{6}\left[-{\gamma_E}^3 - 3{\gamma_E}^2\zeta(2) \; - 2\zeta(3)\right]\epsilon^2 + \cdots \quad (7)$$

$$e^{\epsilon\gamma_E} \; \Gamma[\epsilon] = \frac{1}{\epsilon} + \frac{1}{2}\zeta(2)\epsilon - \frac{1}{3}\zeta(3)\epsilon^2 + \cdots \quad (8)$$

For definitions of Riemann's zeta-numbers $\zeta(n)$ and the Euler constant γ_E see next slides,

N[EulerGamma, 40] =
0.5772156649015328606065120900824024310422

Look at the singularities in the complex plane.
Figure shows the real part of Γ:



$$\begin{aligned}
\text{Gamma}[-1 \pm 10i] &= -4.9974\, 10^{-9} \pm 1.07847\, 10^{-8} i \qquad (9)\\
\text{Gamma}[-1 \pm 100i] &= 1.51438\, 10^{-71} \pm 1.27644\, 10^{-73} i\\
\text{Gamma}[\pm 100.1] &\approx \pm 10^{\pm 157} \qquad (10)
\end{aligned}$$

Just to remind:

$$\zeta(a) = \sum_{k=1}^{\infty} \frac{1}{k^a} \tag{11}$$

$$\text{HarmonicNumber}[N, a] = \sum_{k=1}^{N} \frac{1}{k^a} = H_{N,a} = S_a(N) \tag{12}$$

$$\text{HarmonicNumber}[N] = \sum_{k=1}^{N} \frac{1}{k^1} = H_N = S_1(N) \tag{13}$$

$$\gamma_E = \lim_{N\to\infty} [S_1(N) - \ln(N)] = 0.57721\cdots$$

When using Cauchy's theorem, we will also need derivatives of $\Gamma(z)$:

$$\text{PolyGamma [z]} \equiv \text{PolyGamma [0,z]} = \Psi(z) = \frac{1}{\Gamma(z)} \frac{d}{dz}\Gamma(z) \tag{14}$$

At integer values:

$$\Psi(N+1) \;=\; \sum_{k=1}^{N} \frac{1}{k} - \gamma_E \;=\; S_1(N) - \gamma_E \tag{15}$$

The following properties hold:

$$\Psi(z+1) \;=\; \Psi(z) + 1/z \tag{16}$$

$$\Psi(1+\epsilon) \;=\; -\gamma_E + \zeta_2\,\epsilon + \ldots \tag{17}$$

$$\Psi(1) \;=\; -\gamma_E \tag{18}$$

$$\Psi(2) \;=\; 1 - \gamma_E \tag{19}$$

$$\Psi(3) \;=\; 3/2 - \gamma_E \tag{20}$$

Finally:

$$\mathrm{PolyGamma}[n, z] = \frac{d^n}{dz^n}\Psi(z) \tag{21}$$

It is e.g.

$$\mathrm{PolyGamma}[2N, 1] = -(2N)!\;\;\zeta(2N+1) \tag{22}$$

Cauchy Theorem and Residues

An integral over an anti-clockwise directed closed path C is:

$$\oint F(z)dz = 2\pi i \sum_{z=z_i} \mathrm{Res}[F(z)] \tag{23}$$

where the residues $\mathrm{Res}[F(z)]|_{z=z_i}$ are coefficients a^i_{-1} of the Laurent series of $F(z)$ around z_i:

$$F(z) = \sum_{n=-N}^{\infty} a^i_n (z-z_i)^n = \frac{a^i_{-N}}{(z-z_i)^N} + \cdots + \frac{a^i_{-1}}{(z-z_i)} + a^i_0 + \cdots$$

$$\mathrm{Res}[F(z)]|_{z=z_i} = a^i_{-1} \tag{24}$$

If $G(z)$ has a Taylor expansion around z_0 , then it is:

$$\mathrm{Res}[G(z)\ F(z)]|_{z=z_i} = \sum_{n=1}^{N} \frac{a^i_{-n}}{k!} \frac{d^n}{dz^n} G(z)|_{z=z_i} \tag{25}$$

Due to the property (25), we need for applications not only $\Gamma(z)$, but also its derivatives.

Some residues with $\Gamma(z)$

$$\Psi(z) = \text{PolyGamma}[z] = \text{PolyGamma}[0, z] \tag{26}$$

$$\text{Residue}[\Gamma[z], \{z, -n\}] = \frac{(-1)^n}{n!} \tag{27}$$

$$\text{Residue}[F[z]\Gamma[z], \{z, -n\}] = \frac{(-1)^n}{n!} F[-n] \tag{28}$$

$$\text{Residue}[F[z]\Gamma[z]^2, \{z, -n\}] = \frac{2\text{PolyGamma}[n+1]F[-n] + F'[-n]}{(n!)^2} \tag{29}$$

In the last equation, we used (14):

$$\Psi(z) = \text{PolyGamma}[z] = \text{PolyGamma}[0, z] \tag{30}$$

Some further residues derived with Mathematica:

```
Series[Gamma[z]^2, {z, -3, -1}]
```

$$\frac{1}{36\,(z+3)^2} + \frac{\frac{11}{108} - \frac{\text{EulerGamma}}{18}}{z+3} + O[z+3]^0$$

In[8]:= `Series[Gamma[z - 2] Gamma[z + 5]^2, {z, 2, -1}]`

Out[8]= $$\frac{518\,400}{z-2} + O[z-2]^0$$

In[6]:= `Series[Gamma[z + 2] Gamma[z - 1]^2, {z, -2, -1}]`

Out[6]= $$\frac{1}{36\,(z+2)^3} + \frac{\frac{11}{108} - \frac{\text{EulerGamma}}{12}}{(z+2)^2} + \frac{97 - 132\,\text{EulerGamma} + 54\,\text{EulerGamma}}{432\,(z+2)}$$

In[4]:= `Series[Gamma[z + 2] Gamma[z - 1]^2, {z, 1, -1}]`

Out[4]= $$\frac{2}{(z-1)^2} + \frac{3 - 6\,\text{EulerGamma}}{z-1} + O[z-1]^0$$

Integrals + sums, an example

$$\oint_{-1/3-9i}^{-1/3+9i} dz\Gamma[z] \quad = \quad (-i)\ 3.971730\ 48 - 1.\ x\ 10^{-}16$$

$$2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = (2\pi i)\frac{1-e}{e} \quad = \quad (-i)\ 3.971730\ 6097 \qquad (31)$$

Eq. (31) corresponds to closing the integration contour at ∞ to the left.
While, closing the contour to the right gives another result ...
... of course, or not of course?

$$(-1) * 2\pi i \sum_{n=0}^{0} \frac{(-1)^n}{n!} = (2\pi i) \quad \neq \quad (-i)\ 3.97173060759774304 11 \qquad (32)$$

X(s,∞) = Sum[s^(n) Gamma[n + 1]^3/ (n! Gamma[2 + 2n]), n, 0, Infinity] =
(4*ArcSin[Sqrt[s]/2])/(Sqrt[-s(s-4)])

$$
\begin{aligned}
X_{num}(10 \pm i\epsilon, 1000) &= 4.068032763086749\ x\ 10^{396} \pm 4.06531848\ x\ 10^{389}\ I \\
X_{ana}(10 \pm i\epsilon) &= -0.532777 \pm 0.811156\ I \\
X_{num}(-10, 1000) &= 1.742612203133043 * 10^{396} \\
X_{ana}(-10) &= 0.41884
\end{aligned} \tag{33}
$$

Y(s,∞) = Sum[s^(n) PolyGamma[0, n + 1], n, 0, Infinity] =
(EulerGamma + Log[1 - s])/(-1 + s)

The sums were done with Mathematica v.5.2 and v.11

$$
\sum_{n=0}^{\infty} s^n \frac{\Gamma^3(n+1)}{n!\ \Gamma(2n+2)} = \frac{\arcsin(\sqrt{s/2})}{\sqrt{-s(s-4)}} \tag{34}
$$

$$
\sum_{n=0}^{\infty} s^n \mathrm{PolyGamma}(0, n+1) = \frac{\gamma_E + \log(1-s)}{s-1} \tag{35}
$$

Derivation of the Feynman parameter representation for L-loop n-point Feynman integrals of tensor rank R with N internal lines

L-loop n-point Feynman Integrals of tensor rank R with N internal lines

- Internal loop momenta are $k_l, \quad l = 1 \cdots L$
- Propagators have mass m_i and momentum $q_i, \quad i = 1 \cdots N$ and indices ν_i – see $G(X)$
- External legs have momentum $p_e, \quad e = 1 \cdots n$, with $p_e^2 = M_e^2$

The N propagators are:

$$D_i \quad = \quad q_i^2 - m_i^2 \; = [\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{n} d_i^e p_e]^2 - m_i^2$$

Feynman integrals have the following general form:

$$G(X) \quad = \quad \frac{e^{\epsilon \gamma_E L}}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L \;\; X(k_{l_1}, \cdots, k_{l_R})}{D_1^{\nu_1} \ldots D_i^{\nu_i} \ldots D_N^{\nu_N}} .$$

The numerator X may contain a tensor structure (see later for more on that):

$$X(k_{l_1}, \cdots, k_{l_R}) \; = \; (k_{l_1} P_{e_1}) \cdots (k_{l_R} P_{e_R}) \; = \; (P_{e_1}^{\alpha_1} \cdots P_{e_R}^{\alpha_R}) \, (k_{l_1}^{\alpha_1} \cdots k_{l_R}^{\alpha_R})$$

Tensor integrals

Tensor integrals appear naturally in Feynman diagrams, due to

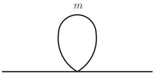
- fermion propagators
- non-abelian triple-boson vertices
- boson propagators in R_ξ gauges and unitary gauge

Example: Fermionic vacuum polarization

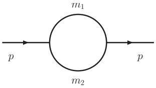
$$\begin{aligned} \Pi^{\alpha\beta} &\sim \frac{1}{(i\pi^{d/2})} \int d^d k \mathrm{Tr} \left[\frac{[\gamma k + m_1]}{D_1} \gamma^\beta \frac{[\gamma(k+p_1) + m_2]}{D_2} \gamma^\alpha \right] \\ &\sim \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2} \left[(m_1 m_2 - k^2 - k p_1) g^{\alpha\beta} + 2 k^\alpha k^\beta \right. \\ &\quad \left. + k^\alpha p_1^\beta + p_1^\alpha k^\beta \right] \end{aligned} \tag{36}$$

So, one needs also efficient ways to evaluate tensor integrals – see later

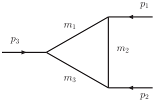
Simple examples of scalar integrals



$$A_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1} \to \mathrm{U}V - divergent : \sim \frac{d^4 k}{k^2}$$



$$B_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2} \to \mathrm{U}V - divergent \sim \frac{d^4 k}{k^4}$$



$$C_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2 D_3} \to \mathrm{U}V - finite \sim \frac{d^4 k}{k^6}$$

Dependent on conventions, where k starts to run in the loop, it is:

$$D_1 = k^2 - m_1^2$$
$$D_2 = (k + p_1)^2 - m_2^2$$
$$D_3 = (k + p_1 + p_2)^2 - m_3^2$$

For a treatment of the UV-divergencies, but also of the infrared- divergencies, we need a regularization method.

Evaluate Feynman integrals

We consider here only Feynman integrals in dimensional regularization

There are several strategies to solve a Feynman integral:

- Reduction

  Express the needed integral with the aid of recurrence relations $\rightarrow$ P. Marquard's lecture by a smaller number of integrals.

  These are then the Master Integrals.
- Direct evaluation of the needed integral(s), with many ways to do this
  - Evaluate the Feynman parameter representations (infrared problems $\rightarrow$ sector decomposition $\rightarrow$ P. Marquard's lecture)
    – analytic or numeric
  - Derive (systems of) differential or difference equations
    – analytic or numeric
  - Derive and solve Mellin-Barnes representations (infrared problems $\rightarrow$ OK)
    – analytic or numeric

Introduction of Feynman parameters

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \ldots D_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \ldots + \nu_N)}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{\nu_1 - 1} \ldots x_N^{\nu_N - 1} \delta(1 - x_1 \ldots - x_N)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}}$$

with $N_\nu = \nu_1 + \ldots \nu_N$.
The denominator of G contains, after introduction of Feynman parameters x_i, the momentum dependent function m^2 with index-exponent N_ν:

$$(m^2)^{-(\nu_1 + \ldots + \nu_N)} = (x_1 D_1 + \ldots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2 Q_j k_j + J)^{-N_\nu} \quad (37)$$

Here M is an (LxL)-matrix, $Q = Q(x_i, p_e)$ an L-vector and $J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l})$.
M, Q, J are linear in x_i. The momentum integration is now simple:
Shift the momenta k such that m^2 has no linear term in $\bar{k}$:

$$\begin{aligned} k &= \bar{k} + (M^{-1}) Q, \\ m^2 &= \bar{k} M \bar{k} - Q M^{-1} Q + J. \end{aligned} \quad (38)$$

Remember: $M_{1-loop} = 1$, in general:

$$M^{-1} = \frac{1}{(\det M)} \tilde{M}, \quad (39)$$

where $\tilde{M}$ is the transposed matrix to M. The shift leaves the integral unchanged.

The shift leaves the integral unchanged (rename $\bar{k} \to k$):

$$G(1) = \int \frac{Dk_1 \dots Dk_L}{\left(kMk + J - QM^{-1}Q\right)^{N_\nu}}. \tag{40}$$

Go Euclidean: Rotate now the $k^0 \to iK_E^0$ with $k^2 \to -k_E^2$ (and again rename $k^E \to k$):

$$G(1) \to (i)^L \int \frac{Dk_1^E \dots Dk_L^E}{\left(-k^E M k^E + J - QM^{-1}Q\right)^{N_\nu}} = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \dots Dk_L}{\left[kMk - (J - QM^{-1}Q)\right]^{N_\nu}}.$$

Call

$$\mu^2(x) = -(J - QM^{-1}Q) \tag{41}$$

and get

$$G(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \dots Dk_L}{\left(kMk + \mu^2\right)^{N_\nu}}. \tag{42}$$

For 1-loop integrals it is $L = 1, M = 1$ - and we will use nearly only those - we are ready to do the k-integration.

Additional step for L-loop integrals

For L-loops go on and now diagonalize the matrix M by a rotation:

$$
\begin{aligned}
k \to k'(x) &= V(x)\, k, \\
kMk &= k' M_{diag} k' \\
&\to \sum \alpha_i(x) k_i^2(x), \\
M_{diag}(x) &= (V^{-1})^{+} M V^{-1} \;=\; (\alpha_1, \ldots, \alpha_L).
\end{aligned}
$$

This leaves both the integration measure and the integral invariant:

$$
G(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{\left(\sum_i \alpha_i k_i^2 + \mu^2\right)^{N_\nu}}. \tag{43}
$$

Rescale now the k_i,

$$\bar{k}_i = \sqrt{\alpha_i} k_i, \tag{44}$$

with

$$d^d k_i = (\alpha_i)^{-d/2} d^d \bar{k}_i, \tag{45}$$

$$\prod_{i=1}^{L} \alpha_i = \det M, \tag{46}$$

and get the Euclidean integral to be calculated (and rename $\bar{k} \to k$):

$$G(1) = (-1)^{N_\nu} (i)^L (\det M)^{-d/2} \int \frac{Dk_1 \ldots Dk_L}{\left(k_1^2 + \ldots + k_L^2 + \mu^2\right)^{N_\nu}}.$$

Use now (remembering that $Dk = dk/(i\pi^{d/2})$):

$$i^L \int \frac{Dk_1 \ldots Dk_L}{\left(k_1^2 + \ldots + k_L^2 + \mu^2\right)^{N_\nu}} = \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma\left(N_\nu\right)} \frac{1}{\left(\mu^2\right)^{N_\nu - dL/2}}, \tag{47}$$

$$i^L \int \frac{Dk_1 \ldots Dk_L \; k_1^2}{\left(k_1^2 + \ldots + k_L^2 + \mu^2\right)^{N_\nu}} = \frac{d}{2} \frac{\Gamma\left(N_\nu - \frac{d}{2}L - 1\right)}{\Gamma\left(N_\nu\right)} \frac{1}{\left(\mu^2\right)^{N_\nu - dL/2 - 1}}.$$

These formulae follow for $L = 1$ immediately from any textbook.
See 'Mathematical Interlude'.

For $L > 1$, get it iteratively, with setting $(k_1^2 + k_2^2 + m^2)^N = (k_1^2 + M^2)^N$, $M^2 = k_2^2 + m^2$, etc.

athematical interlude: d-dimensional 1-loop integrals

After the Wick rotation, the integrand of the momentum integration is positive definite.
Further it is independent of the angular variables.
The integral is understood as symmetric limit the infinity boundaries.

$$\int d^d k \; k_\mu \; F(k^2) = 0$$
$$\int d^d k \; F(k+C) = \int d^d k \; F(k).$$

Introduce d-dim. spherical coordinates. The vector k has d components:

$$\begin{aligned} k_d &= r\cos\theta_d \equiv \rho_d \cos\theta_d \\ k_{n-1} &= \rho_{n-1}\cos\theta_{n-1} \\ & \ldots \\ k_3 &= \rho_3 \cos\theta_3 \\ k_2 &= \rho_2 \sin\phi \\ k_1 &= \rho_2 \cos\phi \\ \rho_{n-1} &= \rho_n \sin\theta_n \end{aligned}$$

The variables are: $\phi, \rho_d, \theta_n (n = 3 \cdots d)$.

Mathematical interlude (II)

The above is the direct generalization of the 3- or 4-dimensional phase space parametrization.
With these variables, the integral over the complete d-dimensional phase space gets the following form:

$$\int_{-\infty}^{\infty} d^d k \; F(k) = \lim_{R\to\infty} \int_0^R dr r^{d-1} \int_0^{\pi} d\theta_{d-1} \sin^{d-2}\theta_{d-1} \int_0^{\pi} d\theta_{d-2} \sin^{d-3}\theta_{d-2} \ldots \int_0^{2\pi} d\theta_1 F(k)$$

The integrations met in the loop calculations may be performed using the following two integrals:

$$\int_0^{\pi} d\theta \sin^m\theta = \sqrt{\pi}\frac{\Gamma\left[\frac{1}{2}(m+1)\right]}{\Gamma\left[\frac{1}{2}(m+2)\right]}, \quad (48)$$

$$\int_0^{\infty} dr \frac{r^\beta}{(r^2+M^2)^\alpha} = \frac{1}{2}\frac{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\alpha-\frac{\beta+1}{2}\right)}{\Gamma(\alpha)}\frac{1}{(M^2)^{\alpha-(\beta+1)/2}}.$$

In general, the angular integrations are influenced by the integrand too. (Remember phase space integrals of bremsstrahlung in *d* dimensions!)

Mathematical interlude (III)

If $F(k) \to F(r),\ r = |k|$, the angular integrations yield the surface of the d-dimensional sphere with radius r:

$$\omega_d(r) = \frac{2\pi^{d/2}}{\Gamma\left[\frac{d}{2}\right]} r^{d-1}. \quad (49)$$

The remaining integration, over r, yields for $F(r) = 1$ the volume of the sphere with radius R:

$$V_d(R) = \frac{\pi^{d/2}}{\Gamma\left[1+\frac{d}{2}\right]} R^d, \quad (50)$$

$$
\begin{aligned}
G(1) &= \int d^d k \frac{1}{(k^2+M^2)^{N_\nu}} \\
&= \int_0^\infty dr \frac{\omega_d(r)}{(r^2+M^2)^{N_\nu}}
\end{aligned}
$$

and we get immediately, with $M^2 \equiv M^2(x_1, x_2, \ldots)$ for 1-loop integrals, $L = 1$:

$$
G(1) = \left[\frac{i\pi^{d/2}\Gamma\left(N_\nu - d/2\right)}{\Gamma(N_\nu)} \frac{1}{(M^2)^{N_\nu - d/2}} \right]. \tag{51}
$$

Finally, one gets for scalar L-loop integrals:

$$G(1) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \, x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{(\det M)^{-d/2}}{(\mu^2)^{N_\nu - dL/2}},$$

or

$$G(1) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \, x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

with

$$U(x) = (\det M) \; (\to 1 \text{ for } L = 1) \quad (52)$$

$$F(x) = (\det M)\, \mu^2 = -(\det M)\, J + Q\, \tilde{M}\, Q \; (\to -J + Q^2 \text{ for } L = 1) \quad (53)$$

Trick for one-loop functions:

$$U = \det M = 1 = \sum x_i \quad (54)$$

and so U ‘disappears’ and the construct $F_1(x)$ is bilinear in $x_i x_j$:

$$F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j. \tag{55}$$

The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand
$k \to \bar{k} + U(x)^{-1}\tilde{M}Q$
the $\int d^d\bar{k}\ \bar{k}/(\bar{k}^2+\mu^2) \to 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j\ x_j^{\nu_j-1} \delta[1-\sum_{i=1}^{N} x_i] \frac{U(x)^{N_\nu - d(L+1)/2-1}}{F(x)^{N_\nu - dL/2}} \left[\sum_l \tilde{M}_{1l} Q_l\right]_\alpha ,$$

Here also a tensor integral:

$$G(k_{1\alpha}k_{2\beta}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu -2- d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$
$$\times \sum_l \left[[\tilde{M}_{1l}Q_l]_\alpha [\tilde{M}_{2l}Q_l]_\beta - \frac{\Gamma\left(N_\nu - \frac{d}{2}L - 1\right)}{\Gamma\left(N_\nu - \frac{d}{2}L\right)} \frac{g_{\alpha\beta}}{2} U(x)F(x) \frac{(V_{1l}^{-1})^{+}(V_{2l}^{-1})}{\alpha_l} \right]$$

The 1-loop case will be used in the following L times for a sequential treatment of an L-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, kp_e]) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \, x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{[1, Qp_e]}{F(x)^{N_\nu - d/2}} \tag{56}$$

Examples for one-loop F-polynomials

One-loop vertex:

$$F(t, m^2) = m^2(x_1 + x_2)^2 + [-t]x_1 x_2$$

one-loop box:

$$F(s, t, m^2) = m^2(x_1 + x_2)^2 + [-t]x_1 x_2 + [-s]x_3 x_4$$

one-loop pentagon:

$$F(s, t, t', v_1, v_2, m^2) = m^2(x_1 + x_3 + x_4)^2 + [-t]x_1 x_3 + [-t']x_1 x_4 + [-s]x_2 x_5 + [-v_1]x_3 x_5 + [-v_2]x_2 x_4$$

2-loop example: B7l4m2 = B2 (page 8), has a box-type sub-loop with 2 off-shell legs: (diagram see next page):

$$F^{-(a_{4567}-d/2)} = \Big\{ m^2(x_5 + x_6)^2 + [-t]x_4 x_7 + [-s]x_5 x_6 + (m^2 - Q_1^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_2^2)x_7 x_5 \Big\}^{-(a_{4567}-d/2)}$$

2-loop: B5l2m2, sub-loop with 2 off-shell legs (diagram see next page):

$$F_{2lines}(k_1^2, m^2) = m^2(x_3)^2 + [-k_1^2 + m^2]x_1 x_3$$

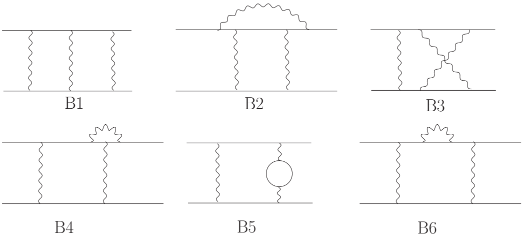


Figure 1: Two-loop box diagrams for massive 2 $\rightarrow$ 2 scattering

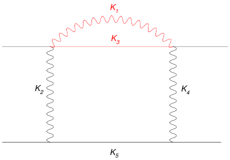
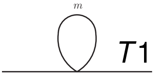


Figure 2: Two-loop box master diagram B5l2m2 (related to B2 = B7l4m2 by shrinking two lines)

The Tadpole $A_0(m)$



$$T1l1m[a] = A_0 \quad = \quad \frac{e^{\epsilon \gamma_E}}{(i\pi^{d/2})} \int \frac{d^d k}{(k^2 - m^2)^a} \to \mathrm{UV-div.}$$

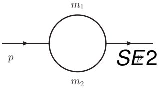
With our general formulae we get, in the 1-dimensional Feynman parameter integral, for the numerator

$$\begin{aligned} N &= (k^2 - m^2)x_1 \equiv k^2 + J \\ F &= m^2 x_1 \equiv m^2 x_1^2 \end{aligned}$$

and thus

$$\begin{aligned} T1l1m[a] &= (-1)^a e^{\epsilon \gamma_E} \frac{\Gamma[a - d/2]}{\Gamma[a]} \int_0^1 dx x^{a-1} \delta[1-x] \frac{1}{F^{a-d/2}} \\ &= (-1)^a e^{\epsilon \gamma_E} (m^2)^{2-a-\epsilon} \frac{\Gamma[a - 2 + \epsilon]}{\Gamma[a]} \\ &\to -e^{\epsilon \gamma_E} \Gamma[-1 + \epsilon] \ \text{ for a} = 1, \ \text{m} = 1 \\ &= \frac{1}{\epsilon} + 1 + \left(1 + \frac{\zeta_2}{2}\right)\epsilon + \left(1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3}\right)\epsilon^2 + \cdots \end{aligned}$$

The Self-energy $B_0(s, m_1, m_2)$



$$SE2l \;=\; B_0[s, m_1, m_2] \;=\; (2\sqrt{\pi}\mu)^{4-d}\,\frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})}\int \frac{d^d k}{[k^2 - m^2][(k+p)^2 - m_2^2]}$$

The $SE2l$ is UV-divergent and the corresponding F-function is ($p^2 = s$):

$$F[s, m_1, m_2] \;=\; m_1^2 x_1^2 + m_2^2 x_2^2 + [-s + m_1^2 - m_2^2]x_1 x_2 \qquad (57)$$

and for special cases:

$$F[s, m_1, 0] \;=\; m_1^2 x_1^2 + [-s + m_1^2]x_1 x_2 \qquad (58)$$

$$F[s, m_1, m_1] \;=\; m_1^2 (x_1 + x_2)^2 + [-s]x_1 x_2 \qquad (59)$$

$$F[s, 0, 0] \;=\; [-s]x_1 x_2 \qquad (60)$$

The 'conventional' Feynman parameter integral is 1-dimensional because $x_2 \equiv 1 - x_1$:

$$F(x) = -sx(1-x) + m_2^2(1-x) + m_1^2 x \equiv -s(x - x_a)(x - x_b) \qquad (61)$$

The result is of logarithmic type for the constant term in ϵ:

$$\begin{aligned} B_0[s,m_1,m_2] &= (4\pi\mu^2)^\epsilon \, e^{\epsilon\gamma_E} \, \frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 \frac{dx}{F(x)^\epsilon} \\ &= \frac{1}{\epsilon} - \int_0^1 dx \ln\left(\frac{F(x)}{4\pi\mu^2}\right) \\ &\quad + \epsilon \left\{ \frac{\zeta_2}{2} + \frac{1}{2}\int_0^1 dx \ln^2\left(\frac{F(x)}{4\pi\mu^2}\right)\right\} + \mathcal{O}(\epsilon^2). \end{aligned}$$

Here we used the expansion:

$$e^{\epsilon\gamma_E}\Gamma(1+\epsilon) = 1 + \frac{\zeta_2}{2}\epsilon^2 - \frac{\zeta_3}{3}\epsilon^3 \cdots \qquad (62)$$

When using `LoopTools`, the corresponding call returns exactly the constant term of B_0 in ϵ (with use of $e^{\epsilon\gamma_E} = 1 + \epsilon\gamma_E + \cdots \to 1$):

$$B_0^{(0)}(s, m_1^2, m_2^2) \quad = \quad \mathtt{b0(s,am12,am22)} \qquad (63)$$

For $4\pi\mu^2 \to 1$ B_0 looks quite compact:

$$B_0(s, m_1, m_2) \quad = \quad \frac{1}{\epsilon} - \int_0^1 dx \; \ln[F(x)] + \frac{\epsilon}{2}\left[\zeta_2 + \int_0^1 dx \ln^2[F(x)]\right] + \cdots \quad (64)$$

Explicitly, one has to integrate

$$\begin{aligned} \ln[F(x)] &= \ln[-s(x-x_a)(x-x_b)] \\ \ln^2[F(x)] &= \ln^2[-s(x-x_a)(x-x_b)] \end{aligned}$$

So we will need the integrals:

$$\int dx_0^1 \left\{\ln(x-x_a),\; ln(x-x_a)ln(x-x_b)\right\} \quad (65)$$

which is trivial, together with some complex algebra rules how to handle complex arguments of logarithms with

$$s \to s + i\epsilon \quad (66)$$

wherever needed.

For the case (59) $m_1 = m_2 = 1$, one gets for the first terms in ϵ:

$$B_0[s,1,1] = \frac{1}{\epsilon} + 2 + \frac{1+y}{1-y} H(0,y), \tag{67}$$

$$H(0,y) = \ln(y). \tag{68}$$

The $H(0,y)$ is a harmonic polylogarithmic function, and

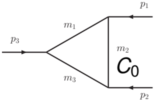
$$y = \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}}$$

$$s = -\frac{(1-y)^2}{y}$$

The other case treated later again is (58) $m_1 = 0, m_2 = m$:

$$B_0[s, m^2, 0] = \frac{1}{\epsilon} + 2 + \frac{1 - s/m^2}{s/m^2} \ln(1 - s/m^2) + O(\epsilon) \tag{69}$$

The massive one-loop vertex $C_0(p_1^2, p_2^2, p_3^2 = s, m_1, m_2, m_3)$



$$= \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})} \int \frac{d^d k}{[(k+p_1)^2 - m^2][k^2][(k-p_2)^2 - m^2]} \sim |_{k\to\infty} \frac{d^4 k}{k^6} \to UV -$$

The massive vertex (all $m_1, m_2, m_3 \neq 0$) is a finite quantity.
But it may be IR-divergent. This we study now.
IR-divergence appears when a massive internal line is between two external on-shell lines.
We assume: $m_2 = 0, m_1 = m_3 = m$ and also: $p_1^2 = p_2^2 = m^2 (on - shell), p_3^2 = s$.
Look at integrand for $k \to 0$:

$$d^4k \ \frac{1}{(k-p_2)^2 - m^2} \ \frac{1}{(k)^2} \ \frac{1}{(k+p_1)^2 - m^2}$$
$$= \ d^4k \ \frac{1}{k^2 - 2kp_2} \ \frac{1}{(k)^2} \ \frac{1}{k^2 + 2kp_1}$$
$$\to \ \frac{d^4k}{k^{1+2+1}} \sim \frac{k^3 dk}{k^4} \sim \frac{dk}{k}|_{k\to 0} \longrightarrow div$$

An IR-regularization is needed, must take $d > 4$ or a small photon mass λ.
If both UV-div (with $d < 4$) and IR-div together: must allow for a complex $d = 4 - 2\epsilon$, and take limit at the end.

First we have a look, for later use, at the F-function:

$$
\begin{aligned}
N &= D_1 x + D_2 y + D_3 z & (71)\\
&= k^2 x + (k^2 + 2kp_1) y + (k^2 - 2kp_2) z & (72)\\
&= k^2 (x + y + z) + 2k(p_1 y - p_2 z) & (73)\\
&= (k + Q)^2 - Q^2 & (74)
\end{aligned}
$$

We used $1 = x + y + z$ here. And the F-function is $F = Q^2 - J = Q^2$ (there is no constant term in N here), as was shown before:

$$
F = m^2 (y + z)^2 + [-s] yz \qquad (75)
$$

This F-function does not factorize in y and z. But now back to the direct Feynman parameter integration.

Start with change $y \to y' = (1-x)y$, then change $x \to (1-x')$:

$$
\begin{aligned}
\frac{1}{D_1 D_2 D_3} &= \int_0^1 dxdydz \frac{\delta(1-x-y-z)}{(D_2 x + D_1 y + D_3 z)^3} \\
&= \int_0^1 dx \int_0^{1-x} \frac{dy}{(D_2 x + D_1 y + D_3(1-x-y))^3} \qquad (76)\\
&= |_{y\to(1-x)y'} \int_0^1 dx \int_0^1 \frac{x\,dy}{(D_2 x + D_1(1-x)y' + D_3(1-x)(1-y'))^3} \\
&= |_{x\to(1-x')} \int_0^1 dx' \int_0^1 \frac{(1-x')dy}{(D_2 x' + D_1 x' y' + D_3 x'(1-y'))^3}
\end{aligned}
$$

After this change of variables, the F-function factorizes in x' and y':

$$
\begin{aligned}
N &= (k + x' p_{y'})^2 - x^2 p_{y'}^2 \qquad (77)\\
&= (k+Q)^2 - Q^2 \qquad (78)
\end{aligned}
$$

resulting into

$$
\begin{aligned}
F = Q^2 &= x'^2 p_{y'}^2 \qquad (79)\\
p_{y'}^2 &= -sy'(1-y') + m^2 \qquad (80)
\end{aligned}
$$

For C_0 we obtain (with $N_\nu = 3$ and $N_\nu - d/2 = 1 + \epsilon$):

$$C_0[s, m, m, 0] = (-1) e^{\epsilon\gamma_E} \Gamma[1+\epsilon] \int_0^1 \frac{dx}{x^{1+2\epsilon}} \int_0^1 \frac{dy}{(p_y^2)^{1-\epsilon}} \tag{81}$$

This is integrable for $\epsilon < 0$, or $d > 4$, or more general: $d \not< 4$.
The x-integral made simple here, but do not expand $1/x^{1+2\epsilon}$!:

$$\int_0^1 \frac{dx}{x^{1+2\epsilon}} = \frac{x^{-2\epsilon}|_0^1}{-2\epsilon} = -\frac{1^{-2\epsilon} - 0^{-2\epsilon}}{2\epsilon} \tag{82}$$

$$= -\frac{1}{2\epsilon} \tag{83}$$

We see that the IR-singularity is an end-point-singularity in Feynman parameter space.
This is the idea of **sector decomposition** and can be formalized [3, 4].

$$-\frac{1}{2\epsilon} \frac{dy}{(p_y^2)^{1-\epsilon}} = -\frac{1}{2\epsilon} \frac{dy}{(p_y^2)} (p_y^2)^\epsilon \tag{84}$$

$$= -\frac{1}{2\epsilon} \frac{dy}{(p_y^2)} e^{\epsilon \ln(p_y^2)} \tag{85}$$

$$= -\frac{1}{2\epsilon} \frac{dy}{(p_y^2)} \left[1 + \epsilon \ln(p_y^2) + \epsilon^2 \ln^2(p_y^2) + \cdots\right] \tag{86}$$

Here I stop this study.

We see that the further integrations proceed quite similar as for the 2-point function, in fact the $p_y^2 = -sy(1-y) + m^2$ is the same building block.
The integrals to be solved now are more general, they include also denominators $1/p_y^2$:

Some integrals

$$\int dy \ln(y - y_0) = (y - y_0)\ln(y - y_0) - y + C \quad (87)$$

$$\int dy \frac{1}{y - y_0} = \ln(y - y_0) + C \quad (88)$$

$$\int dy \frac{\ln(y - y_0)}{y - y_0} = \frac{1}{2}\ln^2(y - y_0) + C \quad (89)$$

Here, often y is real and y_0 is complex. Then no special care about phases is necessary.

$$\int_0^1 \frac{dx}{x - x_0}\left[\ln(x - x_A) - \ln(x_0 - x_A)\right] = Li_2\left(\frac{x_0}{x_0 - x_A}\right) - Li_2\left(\frac{x_0 - 1}{x_0 - x_A}\right) \quad (90)$$

This formula is valid if x_0 is real.

C_0 with a small photon mass λ

In [5, 2], the C_0-integral is treated with a finite photon mass:

$$\int \frac{d^4k}{(k^2-\lambda^2)(k^2+2kp_1)(k^2-2kp_2)} \tag{91}$$

$$= -i\pi^2 \int_0^1 dydx \frac{y}{x^2 p_y^2 + (1-x)\lambda^2} \tag{92}$$

$$= i\pi^2 \int_0^1 dy \left[\frac{1}{2p_y^2} \ln \frac{\lambda^2}{p_y^2} + \mathcal{O}\left(\lambda/\sqrt{p_y^2}\right)\right], \tag{93}$$

It is easy to see from the term $1/(2p_y^2)\ \ln(\lambda^2)$ the correspondence of $1/(d-4)$ and $\log(\lambda^2)$, which is a universal relation in all 1-loop cases.

Now using Mellin-Barnes Representations

Perform the x-integrations by Mellin-Barnes (MB) integrations

Make the result ready for algorithmic analytical and/or numerical evaluation

MBsuite – Computercodes in Mathematica and Fortran/C++:

- AMBRE - Derive formal Mellin-Barnes representations for Feynman integrals in $d = 4 - 2\epsilon$: planar and non-planar $L = 2, 3$-loops [6, 7] (2015)
- MB.m - Find (i) a well-defined MB at some ϵ and (ii) a continuation $\epsilon \to 0$ (iii) and then an expansion around $d = 4$; Evaluate numerically in Euclidean regions [8]
- As a part of MBtools - further Mathematica codes for simplifying representations and for expansions in a small parameter [9]
- MBnumerics - Numerical calculation of MB-integrals in Minkowskian space-time [10] (2016)
- Numerical integrations using CUBA/Cuhre [11]

Integrating the Feynman parameters – get MB-Integrals

We derived the examples:

$$SE2l1m = B_0(s,m,0) = e^{\epsilon\gamma_E}\,\Gamma(\epsilon)\int_0^1 dx_1 dx_2 \frac{\delta(1-x_1-x_2)}{F(x)^{\epsilon}} \quad (94)$$

$$V3l2m = C_0(s,m,m,0) = e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{F(x)^{1+\epsilon}} \quad (95)$$

$$F_{SE2l1m} = m^2 x_1^2 + [-s+m^2]x_1 x_2 \quad (96)$$

$$F_{V3l2m} = m^2(x_1+x_2)^2 + [-s]x_1 x_2 \quad (97)$$

We want to apply now a general formula, valid for all cases:

$$\int_0^1 \prod_{j=1}^{N} dx_j\, x_j^{\alpha_j - 1}\,\delta\left(1-\sum x_i\right) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_N)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_N)} \quad (98)$$

with coefficients α_i dependent on ν_i and on the structure of the F

Eliminate the (+) in (96), (97)

$\rightarrow$ apply one or several MB-integrals here.

Warning: Will not work out naively for all multi-loop cases.

$$\int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\alpha_j - 1} \; \delta\left(1 - \sum_{i=1}^{N} x_i\right) \quad = \quad \frac{\prod_{i=1}^{N} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{N} \alpha_i\right)} \tag{99}$$

Simplest cases, the general one in (99) is easily derived by mathematical method of induction (go from N to $N+1$):

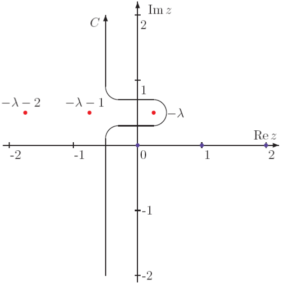
$$\int_0^1 dx_1 \; x_1^{\alpha_1 - 1} \; \delta(1 - x_1) \quad = \quad 1$$

$$\int_0^1 \prod_{j=1}^{2} dx_j \; x_j^{\alpha_j - 1} \; \delta\left(1 - \sum_{i=1}^{N} x_i\right) = \int_0^1 dx_1 x_1^{\alpha_1 - 1}(1 - x_1)^{\alpha_2 - 1} \quad = \quad B(\alpha_1, \alpha_2)$$

$$= \quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$$

Here we want to go:

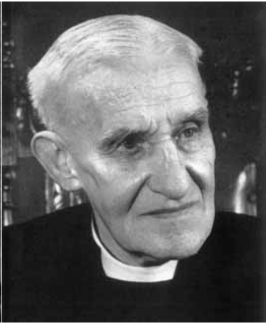
$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}} \quad (100)$$



The integration path separates poles of $\Gamma[\lambda+z]$ and $\Gamma[-z]$.
The formula looks a bit unusual to loop people, but for persons with a mathematical background it is common knowledge.

One might well assume that these two gentlemen did not dream of so heavy use of their results in basic research · · ·

Mellin, Robert, Hjalmar, 1854-1933
Barnes, Ernest, William, 1874-1953

Barnes' contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290
Consider

$$F(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma(-z)^{\sigma}\frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} \qquad (101)$$

where $|\arg(-z)| < \pi$ **(i.e.** $(-z)$ **is not on the neg. real axis) and the path is such that it separates the poles of** $\Gamma(a+\sigma)\Gamma(b+\sigma)$ **from the poles of** $\Gamma(-\sigma)$**.**
$1/\Gamma(c+\sigma)$ **has no pole.**
Assume $a \neq -n$ **and** $b \neq -n, n = 0, 1, 2, \cdots$ **so that the contour can be drawn.**
The poles of $\Gamma(\sigma)$ **are at** $\sigma = -n, n = 1, 2, \cdots$**, and it is:**

```
Residue[ F[s] Gamma[-s] , {s,n} ]  =  (-1)^n/n!  F(n)
```

Closing the path to the right gives then, by Cauchy's theorem, for $|z| < 1$ the hypergeometric function ${}_2F_1(a, b, c, z)$ (for proof see textbook):

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma(-z)^{\sigma}\,\frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} = \sum_{n=0}^{N\to\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\,\frac{z^n}{n!}$$
$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\quad {}_2F_1(a, b, c, z)$$

The continuation of the hypergeometric series for $|z| > 1$ is made using the intermediate formula

$$F(z) = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(1-c+a+n)\sin[(c-a-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(b-a-n)\pi]}(-z)^{-a-n}$$
$$+\sum_{n=0}^{\infty}\frac{\Gamma(b+n)\Gamma(1-c+b+n)\sin[(c-b-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(a-b-n)\pi]}(-z)^{-b-n}$$

and yields

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \quad {}_2F_1(a,b,c,z) \quad = \quad \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a} \quad {}_2F_1(a,1-c+a,1-b+ac,z^{-1})$$
$$+\frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b} \quad {}_2F_1(b,1-c+b,1-a+b,z^{-1})$$

Corollary I

Putting $b = c$, we see that

$$
{}_2F_1(a,b,b,z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!}
$$

$$
= \frac{1}{(1-z)^a} = \frac{1}{2\pi i\, \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma\, (-z)^{\sigma}\, \Gamma(a+\sigma)\Gamma(-\sigma)
$$

This allows to replace sum by product:

$$
\frac{1}{(A+B)^a} = \frac{1}{B^a[1-(-A/B)]^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma A^{\sigma}\, B^{-\sigma-a}\, \Gamma(a+\sigma)\Gamma(-\sigma)
$$

Barnes' lemma

If the path of integration is curved so that the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$ lie on the right of the path and the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ lie on the left, then

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(c-\sigma)\Gamma(d-\sigma) \quad = \quad \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

It is supposed that a, b, c, d are such that no pole of the first set coincides with any pole of the second set.
Scetch of proof: Close contour by semicircle C to the right of imaginary axis. The integral exists and $\int_C$ vanishes when $(a+b+c+d-1)<0$. Take sum of residues of the integrand at poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of Γ-functions, this in turn by combinations of *sin*, may be simplifies finally to the r.h.s.
Analytical continuation: The relation is proved when $(a+b+c+d-1)<0$. Both sides are analytical functions of e.g. a. So the relation remains true for all values of a, b, c, d for which none of the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$, as a function of σ, coincide with any of the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$.

Corollary II Any real shift k: $\sigma+k, a-k, b-k, c+k, d+k$ together with $\int_{-k-i\infty}^{-k+i\infty}$ leaves the result true.

How can the Mellin-Barnes formula be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

$$\frac{1}{(p^2 - m^2)^a} = \frac{1}{2\pi i\, \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}}\, \Gamma(a+\sigma)\Gamma(-\sigma)$$

  which transforms a massive propagator to an integral over massless ones (with index a of the line changed to $(a+\sigma)$).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting F- and U-forms, in order to get products of monomials in the x_i, which allows the integration over the x_i:

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma}\, \Gamma(a+\sigma)\Gamma(-\sigma)$$

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.

A short remark on history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION" [12]:
  A finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals" [13]:
  N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box" [14]:
  Treat UV and IR divergencies in $d = 4 - 2\epsilon$ by analytical continuations: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", [15]:
  Nice algorithmic approach, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined and then going on
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR):
  "Automatized analytic continuation of Mellin-Barnes integrals" [8]:
  Tausk's approach realized in an open-source Mathematica program MB.m, numerics good for many Euklidean cases
- Gluza, Dubovyk, Kajda, Riemann, Usovitsch, 2007-2016 → MB + AMBRE, MBnumerics [6, 7, 10]:
  AMBRE 3 for non-planar diagrams, and MBnumerics good in the Minkowskian

A 1-loop self-energy: SE2l1m

This is a nice example, being simple but showing [nearly] all essentials in a nutshell. We got for this self-energy the F-function (58) with only one "+":

$$F(x) = m^2 x_1^2 + [-s + m^2] x_1 x_2, \tag{102}$$

the following 1-dim. MB-representation (not caring about factors like $e^{\epsilon\gamma_E}$ or $(m^2)^{-\epsilon}$):

$$SE2l1m = \frac{1}{2\pi i} \frac{1}{\Gamma[2-2\epsilon]} \int_{\Re e z = -1/8} dz \left(\frac{[-s+m^2]}{m^2} \right)^{-\epsilon - z} \tag{103}$$

$$\times \Gamma[1-\epsilon-z]\Gamma[-z]\Gamma[1-\epsilon+z]\Gamma[\epsilon+z] \tag{104}$$

Tausk approach:
Seek a configuration where all arguments of Γ-functions (in the numerator) have positive real part. Reasoning $\rightarrow$ see next page!
If found, then the $SE2l1m$ is well-defined and finite.
For small ϵ this is - here - evidently impossible; set $\epsilon \rightarrow 0$ and look at $\Gamma_2[-z]\Gamma_4[+z]$:

$$\Gamma_1[1-z]\Gamma_2[-z]\Gamma_3[1+z]\Gamma_4[+z] \tag{105}$$

What to do ????
Tausk: Set $|\epsilon| \neq$ *small* such that all arguments of Γ-functions **do get** positive real parts, e.g. with the choice:

$$\Re e z = -1/8 \quad \textbf{and} \quad \epsilon = 3/8 \tag{106}$$

To make physics we have now to analytically continue the integrand such that $\epsilon \to 0$; when crossing a residue, take it and add it up.
Varying $\epsilon \to 0$ from 3/8 makes crossing in $\Gamma_4[\epsilon + z]$ a pole at $\epsilon = -z = +1/8$; there is $\epsilon + z = 0$:

$$\textbf{Residue}[\textbf{SE2l1m}, \{z, -\epsilon\}] = \frac{1}{2\pi i\, \Gamma[2-2\epsilon]} \Gamma_3[1-2\epsilon]\Gamma_2[\epsilon] \tag{107}$$

Here we 'loose' one integration (easier term!) and catch the IR-singularity in $\Gamma_2[\epsilon] \sim 1/\epsilon$!
The sum of MB-integral and ($2\pi i$ * Residue) becomes now, for small ϵ:

$$\begin{aligned} \textbf{SE2l1m} \quad = \quad & \frac{1}{2\pi i} \frac{1}{\Gamma[2-2\epsilon]} \int_{\Re z=-1/8} dz \left[\frac{-s+m^2}{m^2}\right]^{-\epsilon-z} \\ & \times \Gamma_1[1-\epsilon-z]\Gamma_2[-z]\Gamma_3[1-\epsilon+z]\Gamma_4[\epsilon+z] \\ & + \\ & \frac{1}{\Gamma[2-2\epsilon]} \Gamma_1[1-2\epsilon]\Gamma_2[\epsilon] \end{aligned}$$

We take safely the limit $\epsilon \to 0$ in both terms; whereever needed (2nd term), make an expension in ϵ (with $\Gamma[2-2\epsilon] \to 1+\epsilon$):

$$\mathbf{SE2l1m} = \frac{1}{2\pi i}\int_{\Re z=-1/8} dz \left[\frac{-s+m^2}{m^2}\right]^{-z} \Gamma_1[1-z]\Gamma_2[-z]\Gamma_3[1+z]\Gamma_4[z] + O(\epsilon)$$
$$+ \left(2+\frac{1}{\epsilon}\right) + O(\epsilon) \qquad (108)$$

Now we close the integration path to the left, catch all residues from $\Gamma_3[1+z]\Gamma_4[z]$ for $z < -1/8$, i.e. at $z = -n,\ n = 1, 2, \ldots$:

$$\mathbf{Res}\left\{\left[\frac{-s+m^2}{m^2}\right]^{-z} \Gamma_1[1-z]\Gamma_2[-z]\Gamma_3[1+z]\Gamma_4[z], \{z, -n\}\right\}$$
$$= (-s+m^2)^n \ln(-s+m^2) \qquad (109)$$

In Mathematica:

```
  Assuming [\{ n [Element] Integers, n >= 1 \},
  Residue  [a^(-z) Gamma[1-z] Gamma[-z] Gamma[1+z] Gamma[z], {z,-n} ]]
=
  (-1)^(-2 n) a^n Log[a] ->            a^n Log[a]
```

The sum to be done is trivial (in this trivial case!!):

$$\sum_{n=1}^{\infty} \left[\frac{-s+m^2}{m^2} \right]^n = \frac{1}{1 - \frac{-s+m^2}{m^2}} - 1 \qquad (110)$$

and we end up with:

$$\textbf{SE2l1m} = \frac{1}{\epsilon} + 2 + \left[\frac{1 - s/m^2}{s/m^2} \ln(1 - s/m^2) \right] + O(\epsilon) \qquad (111)$$

This is what we had also from the direct Feynman parameter integration, see (69).

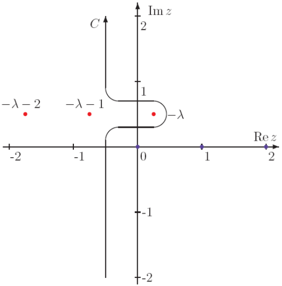
It was not mentioned so far, but is important:
Whenever needed, the causal $i\epsilon$ has to be specified, here:

$$\textbf{SE2l1m} = \frac{1}{\epsilon} + 2 + \frac{m^2 - s + i\epsilon}{s + i\epsilon} \ln \left(1 - \frac{s + i\epsilon}{m^2} \right) + O(\epsilon) \qquad (112)$$

Technical remark on the arguments of the Γ-functions at begin of evaluation of the Mellin-Barnes integrals

We have to ensure that the contour of the Mellin-Barnes representation
– Tausk: straight line parallel to the imaginary axis –
separates the poles of the Γ-functions (numbered here with subindex), see (100):

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} dz \Gamma_1(\lambda+z) \Gamma_2(-z) \frac{B^z}{A^{\lambda+z}} \qquad (113)$$



The integration path separates poles of $\Gamma[\lambda+z]$ and $\Gamma[-z]$.
Further, there is a condition on the arguments of A and B to respect the cuts needed to make everything well-defined.

Look at poles of a one-dimensional MB-integral:

$$\begin{aligned} \Gamma_1: \quad & -z = -N_1, \quad N_1 = 0,1,2,\cdots \\ \Gamma_2: \quad & \lambda + z = -N_2, \quad N_2 = 0,1,2,\cdots \end{aligned} \tag{114}$$

The sequences of pole positions have to be separated by $R = \Re e z$, where z is the integration variable on the contour.
This is possible , for a straight line, only if

$$\Re e \lambda > 0 \tag{115}$$

and if

$$-|\Re e \lambda| < R < 0. \tag{116}$$

Combined condition:

$$-\Re e \lambda < R < 0 \tag{117}$$

This corresponds to:

$$\Re e \lambda + R = \Re e(\lambda + z) \quad > \quad 0 \quad \text{Real part of argument of } \Gamma_1 > 0, \tag{118}$$

$$-R = -\Re e(z) \quad > \quad 0 \quad \text{Real part of argument of } \Gamma_2 > 0. \tag{119}$$

Look at additional 2 Γ-functions of a two-dimensional MB-integral:

Then the second step is derived, with $A_2 = B_2 + B_3 + \cdots$:

$$\frac{1}{2\pi i}\int_{R_1-i\infty}^{R_1+i\infty} dz_1 \frac{A_1^{z_1}}{(B_2+B_3+\cdots)^{z_1+\lambda}}\Gamma_1\Gamma_2$$

$$= \frac{1}{(2\pi i)^2}\int_{R_1-i\infty}^{R_1+i\infty} dz_1 A_1^{z_1}\Gamma_1\Gamma_2 \int_{R_2-i\infty}^{R_2+i\infty} dz_2 \frac{B_1^{z_2}}{(B_2+B_3+\cdots)^{z_2+(z_1+\lambda)}}\Gamma_3[-z_2]\Gamma_4[z_2+(z_1+\lambda)]$$

Again, conditions for the fulfillment of pole separation have to be studied, now for Γ_3 and Γ_4 dependent on z_2.

The poles are at:

$$-z_2 = -N_3, \quad N_3 = 0,1,2,\cdots \tag{122}$$

$$(z_1+\lambda)+z_2 = -N_4, \quad N_4 = 0,1,2,\cdots \tag{123}$$

Again, both series of pole positions

$$z_2 = N_3, \tag{124}$$

$$z_2 = -N_4 - (z_1+\lambda), \tag{125}$$

are separated by the straight contour $z_C = R + I\Im mz$ if also the arguments of the next 2 Γ-functions have positive real parts.

I do not show the details, it is easily seen.

For more dimensions of the MB-integral, iterate the argument.

A 1-loop vertex: V3l2m

The Feynman integral V3l2m is the QED one-loop vertex function, which is no master. It is infrared-divergent (see this by counting of powers of loop integration momentum k or know it from: massless line between two external on-shell lines)

$$F = m^2(x_1 + x_2)^2 + [-s]x_1 x_2 \tag{126}$$

We will also use $m^2 = 1$ and the the variable

$$y = \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}} \tag{127}$$

$$\begin{aligned} \texttt{V3l2m} &= -\frac{e^{\epsilon\gamma_E}\Gamma(-2\epsilon)}{2\pi i} \int dz_{\Re z=-1/2} (-s)^{-\epsilon-1-z} \frac{\Gamma^2(-\epsilon-z)\Gamma(-z)\Gamma(1+\epsilon+z)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon-2z)} \\ &= \frac{\texttt{V3l2m}[-1]}{\epsilon} + \texttt{V3l2m}[0] + \epsilon\, \texttt{V3l2m}[1] + \cdots . \end{aligned}$$

One may slightly shift the contour by $(-\epsilon)$ and then close the path to the left and get residua from (and only from) $\Gamma(1+z)$: [16]:

$$
\begin{aligned}
V(s) &= -\frac{1}{2s\epsilon}\frac{e^{\epsilon\gamma_E}}{2\pi i}\int\limits_{-i\infty-1/2}^{-i\infty-1/2} dz\ (-s)^{-z}\frac{\Gamma^2(-z)\Gamma(-z+\epsilon)\Gamma(1+z)}{\Gamma(-2z)} \\
&= +\frac{e^{\epsilon\gamma_E}}{2\epsilon}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}\frac{\Gamma(n+1+\epsilon)}{\Gamma(n+1)}.
\end{aligned}
$$

This series may be summed directly with Mathematica![1], and the vertex becomes:

$$
V(s) = +\frac{e^{\epsilon\gamma_E}}{2\epsilon}\Gamma(1+\epsilon)\ {}_2F_1\left[1, 1+\epsilon; 3/2; s/4\right]. \tag{128}
$$

Alternatively, one may derive the ϵ-expansion by exploiting the well-known relation with harmonic numbers $S_k(n) = \sum_{i=1}^{n} 1/i^k$:

$$
\frac{\Gamma(n+a\epsilon)}{\Gamma(n)} = \Gamma(1+a\epsilon)\exp\left[-\sum_{k=1}^{\infty}\frac{(-a\epsilon)^k}{k}S_k(n-1)\right]. \tag{129}
$$

The product $\exp(\epsilon\gamma_E)\Gamma(1+\epsilon) = 1 + \frac{1}{2}\zeta[2]\epsilon^2 + O(\epsilon^3)$ yields expressions with zeta numbers $\zeta[n]$, and, taking all terms together, one gets a collection of inverse binomial sums[2]; the first of them is the IR divergent part:

$$V(s) = \frac{V_{-1}(s)}{\epsilon} + V_0(s) + \cdots \tag{130}$$

$$V_{-1}(s) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{1}{2}\frac{4\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}} = \frac{y}{y^2-1}\ln(y). \tag{131}$$

[1]The expression for $V(s)$ was also derived in [17]; see additionally [18].

[2]For the first four terms of the ϵ-expansion in terms of inverse binomial sums or of polylogarithmic functions, see [16].

The constant term:

$$\texttt{V3l2m[0]} = \frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-s)^{-1-r}\,\frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r]}$$

$$\frac{1}{2}\left[\gamma_E - \ln(-s) + 2\Psi[-2r] - 2\Psi[-r] + \Psi[1+r]\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}S_1(n),$$

There is also the opportunity to evaluate the MB-integrals numerically by following with e.g. a Fortran routine the straight contour.
This applies after the ϵ-expansion.

$\int_{-5i+z}^{+5i+z}$ is usually sufficient.

But: This works fast and stable for Euclidean kinematics where $-s > 0$.

and the ϵ-term:

$$
\begin{aligned}
\texttt{V3l2m[1]} &= \frac{1/4}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-s)^{-1-r}\,\frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r)]} \\
&\Big[\gamma_E^2 + Log[-s]^2 + Log[-s](-2\gamma_E - 4\Psi[-2z] + 4\Psi[-z] - 2\Psi[1+z]) \\
&+\gamma_E(4\Psi[-2z] - 4\Psi[-z] + 2\Psi[1+z]) \\
&-4\Psi[1,-2z] + 2\Psi[1,-z] + \Psi[1,1+z] \\
&+4(\Psi[-2z]^2 - 2\Psi[-2z]\Psi[-z] + \Psi[-z]^2 + \Psi[-2z]\Psi[1+z] \\
&-\Psi[-z]\Psi[1+z]) + \Psi[1+z]^2\Big] \\
&= \frac{1}{4}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}\left[S_1(n)^2 + \zeta_2 - S_2(n)\right].
\end{aligned}
$$

Here, $\Psi[r] = ...$ and $\Psi[1,r] = ...$, and the harmonic numbers $S_k(n)$ are

$$
S_k(n) = \sum_{i=1}^{n}\frac{1}{i^k}, \qquad (132)
$$

The sums appearing above may be obtained from sums listed in Table 1 of Appendix D in [16, 19]:

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} &= \frac{y}{y^2-1} 2\ln(y), \\
\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n) &= \frac{y}{y^2-1}\left[-4\mathrm{Li}_2(-y) - 4\ln(y)\ln(1+y) + \ln^2(y) - 2\zeta_2\right], \\
\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n)^2 &= \frac{y}{y^2-1}\Bigg[16 S_{1,2}(-y) - 8\mathrm{Li}_3(-y) + 16\mathrm{Li}_2(-y)\ln(1+y) \\
&\quad +8\ln^2(1+y)\ln(y) - 4\ln(1+y)\ln^2(y) + \frac{1}{3}\ln^3(y) \qquad (133) \\
&\quad +8\zeta_2\ln(1+y) - 4\zeta_2\ln(y) - 8\zeta_3\Bigg], \\
\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_2(n) &= -\frac{y}{3(y^2-1)}\ln^3(y),
\end{aligned}
$$

Expansion in a small parameter: vertex V3l2m for m^2/s

Use as an example for determining the small mass expansion:

$$\text{V3coefm1} = \text{Coefficient}[\text{V3l2m}[[1,1]], \epsilon, -1] \tag{134}$$

$$= -\frac{1}{2s}\frac{1}{2\pi i}\int_{-i\infty-1/2}^{+i\infty-1/2} dz \left(-\frac{m^2}{s}\right)^z \frac{\Gamma_1[-z]^3\Gamma_2[1+z]}{\Gamma_3[-2z]} \tag{135}$$

If $|m^2/s| << 1$, then the smallest [positive] power of it gives the biggest contribution: its exponent has to be positive and small.

So, close the contour to the right (positive z), and leading terms come from the residua expansion due to poles of $\Gamma_1[-z]^3$ at $z = -1, -2, \cdots$. The residues are terms of a binomial sum:

$$Residue[n] = +\frac{1}{s}\left(\frac{m^2}{s}\right)^n \frac{(2n)!}{(n!)^2}\left[2HarmonicNumber[n] - 2HarmonicNumber[2n] - \ln\left(-\frac{m^2}{s}\right)\right]$$

with first terms equal to (-1)*Residua:

$$V3l2m = \frac{1}{s}\left[\ln\left(-\frac{m^2}{s}\right) + \frac{m^2}{s}\left(2 + 2\ln\left(-\frac{m^2}{s}\right)\right) + \frac{m^4}{s^2}\left(7 + 6\ln\left(-\frac{m^2}{s}\right)\right)\right] + O(m^6/s^4)$$

No summary. One might include remarks on:

- A two-dimensional example, e.g. the massive one-loop box
- Numerical integrations for the case of Minkowskian kinematics:
  perform shifts and deformations of contours, variable transformations, etc. [20, 21, 22]
- A planar two-loop diagram, e.g. a sunrise 2-point function, and a non-planar one, e.g. the non-planar massless double box.

The two-loop examples can be calculated in two different ways:

- Do the derivation of the MB-representation **loop by loop**, so that formally only one-loop properties appear, e.g. $U = 1$. [23]
- Do the calculation with direct F, U-functions [24, 25].
  One has to introduce so-called Cheng-Wu variables; other wise one may meet $\Gamma[0]$ which is not well-defined. The number of MB-dimensions depends.

Citations follow, but: end of lectures

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